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Single strategy funds of hedge funds

This paper provides a simple method to predict how the higher order return properties of a single strategy fund of hedge funds will vary as one more fund is added to, or removed from, the portfolio. Our model-free approach uses average co-moments obtained from the universe of available funds to develop a functional relationship between portfolio return distributions and the number of funds in the portfolio. We show that some single strategy funds of hedge funds may be under-diversified and that covariance, coskewness, and cokurtosis, rather than variance, skewness and kurtosis, matter most in portfolio diversification.

**JEL Classification:** G11, G12, G23.

**Keywords:** hedge funds, asset allocation, diversification, skewness, kurtosis.
1 Introduction

Single strategy funds of hedge funds allocate up to 100% of their capital to a portfolio of funds following a specific hedge fund strategy. While the actual number of funds held by these focused-funds of hedge funds varies widely, there is strong anecdotal evidence that they may not be sufficiently diversified within a mean-variance framework. We show, however, that this apparent under-diversification may, in fact, be optimal when return distribution properties are considered within a four moment framework. To shed light on this, we present a simple method to predict how the return distribution properties will vary as one more fund is added to, or removed from, the fund of hedge funds. Specifically, we extend the decomposition methodology employed by Elton and Gruber (1977) and Conine and Tamarkin (1981) to the mean-variance-skewness-kurtosis environment and use this methodology to show precisely how the impact of diversification depends on each of the moments’ constituent “co-moments” (e.g. covariance).

The number of funds held by a fund of hedge funds depends on its total asset value and on whether it is single-or multi-strategy oriented. Typically, focused-funds of hedge funds allocate capital to 10–15 funds while some large multi-strategy funds may allocate capital to 40–60 funds. Rostron (2000) reports that funds of hedge funds allocate capital to anywhere between six to thirty managers (which may manage more than one fund). Liang (2000) reports that the median asset value in the Hedge Fund Research database is $20 million – this, combined with the fact that the minimum investment requirement of hedge funds typically ranges between $250,000 and $1,000,000, suggests that there are significant barriers to diversification in a fund of hedge funds.

The industry’s high attrition rate and the closure of most funds with good track records to new capital make it difficult to select fund managers and thereby form a large diversified hedge fund portfolio. The deterioration of skew from diversification is one reason managers choose to hold a relatively small number of funds in their portfolios. Poor diversification might also be a result of how fund manager compensation is designed. Brown et al. (2002) argue that the more diversified a fund is, the more likely that an investor will incur an incentive fee regardless of overall fund
performance. Investors, aware of this, avoid investing in well-diversified funds of hedge funds.

It is now well-established that hedge fund returns exhibit significant skew and kurtosis. In part, this is because of hedge funds’ frequent employment of dynamic trading strategies based on highly leveraged positions. If we assume that investors value these higher moments (i.e., if investor utility functions are of higher order that quadratic), then mean-variance portfolio analysis is inadequate and must be extended.

Portfolio selection within a mean-variance-skewness framework has been considered by Lai (1991), Chunhachinda et al. (1997), Sun and Yan (2003), Prakash et al. (2003), and others. Only recently, however, have attempts been made to extend the analysis to a four-moment framework (see, for example, Guidolin and Timmermann (2002), Malevergne and Sornette (2002), and Harvey et al. (2003)). As these papers make clear, this extension is non-trivial and often requires that asset returns follow an assumed data generating process. Despite advances in empirical and theoretical characterizations of asset return distributions, the commonly-used methodologies are susceptible to “model risk” created by an institution’s dependence on its own risk projections. These problems are magnified when the methodologies are applied in the context of hedge fund portfolios. In fact, no model has been shown to be capable of combining the widely varying return distributions of different hedge fund styles in fund of hedge fund portfolio analysis (Polyn (2003)). In addition, models that include the fourth moment typically suffer from a lack of intuitiveness and applicability, and do not investigate the interaction between assets, i.e. co-moments. Our paper overcomes some of these problems by using a model-free framework and by providing an tractable decomposition that can be easily adapted for the specific needs of fund managers.

The remainder of the paper is organized as follows. The next section discusses the data. Section 3 decomposes the first four moments of the portfolio return distribution as a function of the number of funds and verifies the decomposition’s small sample properties. Section 4 provides intuition about the relative contribution of the return comoments. Section 5 concludes. The procedure for unsmoothing the data, derivations, and the bootstrap procedure are outlined in the appendix.
2 Strategy Classification and Data

Hedge fund investment strategies tend to be quite different from the strategies followed by traditional money managers. In principle every fund follows its own proprietary strategy, which means that hedge funds are a very heterogeneous group. It is, however, customary to ask hedge fund managers to classify themselves into one of a number of different strategy groups depending on the main type of strategy followed. We concentrate on the following nine main classes of funds and report their estimated market share in terms of assets under management as found in the June 2002 TASS asset flows report:

Long/Short Equity (43%): Funds that invest on both the long and the short side of the equity market. Unlike equity market neutral funds (see below), the portfolio may not always have zero market risk. Most funds have a long bias.

Equity Market Neutral (7%): Funds that simultaneously take long and short positions of the same size within the same market, i.e. portfolios are designed to have zero market risk. Leverage is often applied to enhance returns.

Convertible Arbitrage (9%): Funds that buy undervalued convertible securities, while hedging (most of) the intrinsic risks.

Distressed Securities (11%): Funds that trade the securities of companies in reorganization and/or bankruptcy, ranging from senior secured debt to common stock.

Merger Arbitrage (8%): Funds that trade the stocks of companies involved in a merger or acquisition, buying the stocks of the company being acquired while shorting the stocks of its acquirer.

Global Macro (9%): Funds that aim to profit from major economic trends and events in the global economy, typically large currency and interest rate shifts. These funds make extensive use of leverage and derivatives. These are the funds that are responsible for most media attention.

Emerging Markets (3%): Funds that focus on emerging and less mature markets. These funds tend to be long only because in many emerging markets short selling is not permitted and futures
and options are not available.

**Dedicated Short Bias (0.23%)**: Funds that invest mostly in short positions in equities and equity derivative products. To effect the short sale, the manager borrows the stock from a counter party (often its prime broker) and sells it in the market.

So-called “funds of funds” are a separate class of funds that solely invest in other hedge funds. Some limit themselves to a specific hedge fund strategy but most invest across the board. The idea behind funds of funds is to offer investors a hassle-free alternative to constructing a basket of hedge funds themselves. In addition, many claim to add value by employing experienced managers to select funds, to carry out due diligence and to monitor continuously the portfolio.

The database used in this study covers the period June 1994–May 2001 and was obtained from Tremont TASS, which is one of the best known and largest hedge fund databases currently available. Our database includes the Asian, Russian and LTCM crises as well as the end of the IT bubble and part of the bear market that followed. As of May 2001, the database contains monthly net of fee returns on a total of 2183 hedge funds and funds of funds. Reflecting the tremendous growth of the industry as well as a notoriously high attrition rate, only 264 of these funds had seven or more years of data available.

Amin and Kat (2003) show that concentrating on only surviving funds overestimates the mean return on individual hedge funds by around 2% as well as introduces biases in estimates of the standard deviation, skewness and kurtosis. To avoid this problem we do not work with the raw return series of the 264 survivor funds but instead to create 358 seven-year monthly return series by, starting off with the 358 funds that were alive in June 1994, replacing every fund that closed down during the sample period by a fund randomly selected from the set of funds alive at the time of closure following the same type of strategy and of similar size and age. Funds of funds (103 funds as of June 1994) are treated in the same way.

This replacement procedure implicitly assumes that in case of fund closure investors are able to roll from one fund into the other at the reported end-of-month net asset values and at zero
additional costs. This underestimates the true costs of fund closure to the investor. First, when a fund closes shop its investors have to look for a replacement investment. This search takes time and is not without costs. Second, investors may get out of the old and into the new fund at values that are less favorable than the end-of-month net asset values contained in the database.

As hedge funds frequently invest in illiquid exchange-traded assets or difficult-to-price over-the-counter securities (Asness, et al. (2001)), a hedge fund manager can have greater discretion in marking the portfolio’s value at the end of each month to arrive at the fund’s net asset value. Hedge funds’ compensation scheme, especially the “high watermark” provision, gives managers an incentive to “smooth” their returns by marking their portfolio to less than their actual value in months with large positive returns so as to create a “cushion” for those months with lower returns.

Managers have difficulty obtaining an accurate value of illiquid assets and most rely on observed transaction prices for similar assets. Similar to its effects in real estate property value, such partial adjustment or “smoothing” produces systematic valuation errors which tend not to be diversified away. It results in serial correlation in hedge fund returns and underestimation of their true standard deviations. We follow the approach of Brooks and Kat (2001), outlined in Appendix A, to unsmooth hedge fund returns and thereby reconcile stale price problems. Table 1 provides a statistical summary of the reported and “unsmoothed” returns. Note that over the sample period, mean monthly returns differ widely across strategies; ranging from $-0.26\%$ for dedicated short bias to $1.37\%$ for long/short equity.

3 Decomposition

Elton and Gruber (1977) analytically express the relation between variance and portfolio size to observe the effects on risk from introducing new securities into the population of securities under study. In a similar manner, we decompose the first four moments of portfolio returns into functions of portfolio size and the population co-moments. Our decomposition formulae can be conveniently integrated into an utility analysis and can substantially facilitate the optimization process by virtue
of their parsimonious, model-free nature.

We assume portfolios are equally-weighted, which is optimal if investors have no information about the future return variance, covariance, coskewness, and cokurtosis. In the case that future co-moments can be forecasted, equal investment is an upper limit on the risk investors bear (Johnson and Shannon (1974)). The total number of hedge funds available within a strategy-class is given by \( N \) and the total number of hedge funds in the portfolio is \( n \).

The variance of a portfolio with \( n \) funds can be decomposed as:

\[
\sigma^2(n) = n^{-2} \left( \sum_{i=1}^{n} \text{var}(i) + \sum_{j=1}^{n} \sum_{i=1, i \neq j}^{n} \text{cov}(i, j) \right).
\]

To see how this might change as \( n \) varies, we first calculate the average variance and average covariance for the total population of available funds \((N >> n)\) within a particular strategy class:

\[
\overline{\text{var}(i)} = \frac{1}{N} \sum_{i=1}^{N} \text{var}(i); \quad \overline{\text{cov}(i, j)} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} \text{cov}(i, j),
\]

where the overline indicates averages over the total available population of funds and the notation \( \text{cov}(i, j) \) implicitly assumes \( i \) and \( j \) are different funds. Substituting these population estimates into (1) and re-arranging, we now assert, and later verify, that a reasonable approximation of the expected standard deviation of a portfolio with \( n \) funds is:

\[
E[\sigma(n)] \approx n^{-1/2} \left[ \overline{\text{var}(i)} + (n-1) \overline{\text{cov}(i, j)} \right]^{1/2}.
\]

Appendix B provides further details of the assumptions underlying this approach. By similar reasoning, we have:

\[
E[\text{skew}(n)] \approx n^{-2} \overline{\text{var}(i)}^{-\frac{3}{2}} \left[ \text{skew}(i) + 3(n-1) \text{coskew}(i, i, k) + (n-1)(n-2) \text{coskew}(i, j, k) \right],
\]

\[
E[\text{kurt}(n)] \approx n^{-3} \overline{\text{var}(i)}^{-2} \left[ \text{kurt}(i) + 4(n-1) \text{cokurt}(i, i, i, k) + 6(n-1)(n-2) \text{cokurt}(i, i, k, l) 
+ (n-1)(n-2)(n-3) \text{cokurt}(i, j, k, l) + 3(n-1) \text{cokurt}(i, i, k, k) \right].
\]

where \( E[\text{skew}(n)] \) is expected relative skewness and \( E[\text{kurt}(n)] \) is expected relative kurtosis of a portfolio with \( n \) funds. We use relative skewness and relative kurtosis since the third and
fourth central moments are scale sensitive and are not amenable to significance testing (see Beedles (1979)). Analogous to our definition of covariance, coskewness is defined as 
\[ \text{coskew}(i, j, k) = E[(R_i - E(R_i)) (R_j - E(R_j)) (R_k - E(R_k))] \]
and cokurtosis is defined as 
\[ \text{cokurt}(i, j, k, l) = E[(R_i - E(R_i)) (R_j - E(R_j)) (R_k - E(R_k)) (R_l - E(R_l))] \]
where \( i, j, k, l \) are restricted to be different funds unless explicitly set the same.

### 3.1 Bootstrap confidence intervals

We now investigate how accurate our decomposition formulae are for portfolios with relatively few hedge funds. The unbiased estimates of the mean (\( \bar{R}_p(n) \)), variance (\( \sigma_p^2(n) \)), skewness (\( M_p(n) \)), and kurtosis (\( K_p(n) \)) based on the returns (\( R_{pt}(n, t) \), \( t = 1, \ldots, T \)) of randomly drawn portfolios of size \( n \) are:
\[
\bar{R}_p(n) = \sum_{t=1}^{T} R_{pt}(n), \tag{5}
\]
\[
\sigma_p^2(n) = \frac{1}{T-1} \sum_{t=1}^{T} (R_{pt}(n) - \bar{R}_p(n))^2, \tag{6}
\]
\[
M_p(n) = \frac{T}{(T-1)(T-2)} \sum_{t=1}^{T} \left( \frac{(R_{pt}(n) - \bar{R}_p(n))^3}{\sigma_p^3(n)} \right), \tag{7}
\]
\[
K_p(n) = \frac{T(T-1)}{(T-1)(T-2)(T-3)} \sum_{t=1}^{T} \left( \frac{(R_{pt}(n) - \bar{R}_p(n))^4}{\sigma_p^4(n)} \right) - \frac{3(T-1)^2}{(T-2)(T-3)}. \tag{8}
\]
Based on these unbiased estimates, we construct bootstrap confidence intervals for the standard deviation, skewness, and kurtosis. We then compare these confidence intervals with the estimates provided by our decomposition formulae, based on the same bootstrap total population sample. The bootstrap simulation procedure is described in Appendix C.

Encouragingly, we find that the moment estimates based on our decomposition formulae for standard deviation and skewness lie within these bootstrap confidence intervals. Representative examples can be seen in Figures 1–2. This suggests that our estimates of standard deviation and skewness, based on the number of funds in the portfolio, provide, on average, an estimate no worse than the unbiased estimates calculated using actual portfolio returns.
The results for kurtosis are less promising. This is expected since the complex interactions of higher moments are much more difficult to model. Except for global macro, long/short equity, and emerging market fund portfolios, all other strategy-focused portfolios have expected kurtosis values based on the decomposition formula (4) that lie outside the 99% confidence bound when the number of funds included is small. Representative examples can be seen in Figure 3.

To correct for the estimation errors in (2)–(4) with small portfolios, we use a polynomial function to fit the bootstrap simulation values. For example, for expected portfolio variance we can rearrange (2) as:

\[
E \left[ \sigma^2(n) \right] = n^{-1} \overline{\text{var}(i)} + (n - 1)n^{-1} \overline{\text{cov}(i,j)} = \overline{\text{cov}(i,j)} + n^{-1}(\overline{\text{var}(i)} - \overline{\text{cov}(i,j)}).
\]

This function can be thought of as the regression model \( Y_n = \alpha + \beta X_n + \varepsilon \), where \( Y_n \) is the expected variance of a randomly drawn portfolio of \( n \) funds; \( X_n = n^{-1} \); \( \alpha \) and \( \beta \) are the fitted coefficients; and \( \varepsilon \) is the residual term. After estimation, we set the coefficients equal to their corresponding population averages: \( \alpha = \overline{\text{cov}(i,j)} \) and \( \beta = (\overline{\text{var}(i)} - \overline{\text{cov}(i,j)}) \). The new values of \( \overline{\text{cov}(i,j)} \) and \( \overline{\text{var}(i)} \) then substitute for their true values when estimating our decomposition for portfolios with few funds (\( n < 10 \)). Small sample versions of (2) and (3) are uniquely identified using this approach. Estimating the kurtosis decomposition formula with a polynomial does not uniquely identify the contribution of \( \text{cokurt}(i, i, i, k) \) and \( \text{cokurt}(i, i, k, k) \) since they are both of the same order of \( n \). Instead, we take a linear combination of them: \( 4\text{cokurt}(i, i, i, k) + 3\text{cokurt}(i, i, k, k) \).

### 4 Relative importance of co-moments

Table 2 provides the bootstrap simulation averages of the components of our decomposition approximations of expected standard deviation and expected relative skewness.\(^1\) Comparing columns 2 and 3 of table 2 shows that our decomposition implies that expected standard deviations can be

\(^1\)The slight differences between comparable numbers in table 1 and 2 are because the bootstrap method minimizes the impact of a single large outlier that may have had a disproportionate influence on the table 1 results.
substantially reduced by diversification within the fund of hedge funds. For example, a single convertible arbitrage fund has an average expected standard deviation of 4.21 whereas a portfolio of a large number of convertible arbitrage funds could have a standard deviation as low as 2.11.

In contrast, comparing the last two columns of table 2 shows that for almost all strategies, a portfolio of funds has a lower expected skewness than a single fund within that strategy group. This is consistent with the theoretical work of Conine and Tamarkin (1981) on the disincentive of diversification through the decomposition of portfolio third central moment. Unlike previous work, however, our approach provides a quantitative measure of how skewness changes as more funds are added. For example, suppose a fund of hedge funds manager has a portfolio of ten merger arbitrage funds and would like to know how much the expected skewness of his portfolio will fall if another fund is added. Using our skewness decomposition, we observe that:

\[
E[\text{skew}(10)] = \frac{1}{100} \times 2 \times 1.4^{-3/2} \times [-4.01 + 3(9)(-2.67) + (9)(8)(-4.53)] = -1.2849
\]

\[
E[\text{skew}(11)] = \frac{1}{121} \times 2 \times 1.4^{-3/2} \times [-4.01 + 3(10)(-2.67) + (10)(9)(-4.53)] = -1.2983
\]

Thus, an additional fund would cause the expected skewness to fall from -1.2849 to -1.2983. The fund manager must balance this deterioration in skewness and the expected cost of managing/purchasing an additional fund against the potential benefits in terms of reduced standard deviation and kurtosis.

In our numerical example, the largest portion of the fall in skewness came from the \( \text{coskew}(i, j, k) \) term. Figures 4 illustrates graphically how the relative importance of each co-moment varies as the number of funds \( n \) changes. The important feature of our decomposition formula (3) is that the terms have different orders with respect to \( n \): specifically, \( O(n^{-2}) \), \( O(n^{-1}) \), and \( O(1) \). Thus, as the number of funds becomes sufficiently large, the first two terms becoming relatively less important, and expected skewness depends only on the average coskewness between three different funds selected from the total population. Figure 5 presents the analogous results for kurtosis. Again, as the number of funds in the portfolio grows sufficiently large, expected standard deviation depends only on covariance and expected kurtosis depends only on the cokurtosis between four different funds (i.e., \( \text{cokurt}(i, j, k, l) \)).
Hedge fund strategies that have higher market exposure (i.e. exposure to common market risk factors), such as global macro, long/short equity, emerging market, and dedicated short bias, exhibit higher average covariance, which is consistent with high correlation across funds in the strategies. Strategies less prone to market risks, such as convertible arbitrage and merger arbitrage, have lower average covariance which is consistent with lower correlation across funds. But these strategies bear another kind of systematic risk: systematic credit risk, which creates moderate correlation within funds. Equity market neutral funds have the lowest average covariance, as they are designed to bear no systematic risk – market and industry exposure are neutralized through “offsetting” long and short positions.

The standard deviation for a portfolio drawn from all funds approaches 2.56 as $n \to \infty$, which is much lower than the standard deviation of a typical fund of hedge funds return, which is expected to be 4.05. Thus, most funds of hedge funds appear to be underdiversified relative to the potential standard deviation gains from diversification “available”. For most strategy classes, as more funds are included into a focused fund of hedge funds, portfolio standard deviation and skewness fall. This reduction of both risk and skewness occurs at a decreasing rate as the number of funds in the portfolio increases. Risk reduction from diversification within the same hedge fund strategy can be achieved by including 20 or more funds. In contrast, skewness continues to fall significantly as the number of funds in the portfolio approaches 30. This suggests that finding the correct cut-off point is very important since too much diversification may be reducing the funds positive skewness, but not be adding much in terms of risk reduction!

Examining Tables 2 and 3 in tandem indicate that diversification deteriorates skew and improves kurtosis in most strategies. Furthermore, strategies with negative expected skewness often also have high kurtosis (leptokurtosis), suggesting a high probability of an extreme event. Merger arbitrage and distressed securities are notable examples of this. Merger arbitrage funds attempt to profit from the spread between the bid price from the acquirer and the trading price of the target after the deal is announced – thus, if the deal fails, the fund could pay a substantial premium. Distressed securities funds face significant losses if companies are downgraded, go into default or
declare bankruptcy. Agarwal and Naik (2004) classify these arbitrage strategies as “short option” strategies which have similar payoffs to writing an out-of-money put on the Russell 3000 index. Convertible arbitrage funds also have a “short option” return profile: the upside potential of the trade is typically known with precision, but the downside cannot be determined with certainty. Global macro and dedicated short bias funds have positive coskewness, which can be attributed to their “long option” trading nature.

The low coskewness and high cokurtosis present in convertible arbitrage, distressed securities and merger arbitrage funds can be also attributed to the relatively higher credit risk they bear. Anson (2002) argues that credit risk distributions generally are left-skewed and fat-tailed. The various event risks such as redemption risk and bankruptcy risk they bear could also affect the skewness and kurtosis of individual fund returns.

The economic interpretation behind the negative coskewness, compounded by positive cokurtosis, may be explained by the “leverage effect” (Black (1976) and Christie(1982)). Merger arbitrage, convertible arbitrage, and distressed securities funds all apply considerable leverage in their investment strategies. A drop in prices can raise operating and financial leverage and hence the volatility of subsequent returns. This is analogous to the “asymmetric volatility” found by Chen, et al. (2001) in stock returns. As all the funds in the same strategy are threatened by the imminent financial distress, they tend be more correlated in terms of extreme events.

An alternative explanation is the “volatility feedback” mechanism (Pindyck (1984) and French et al. (1987)), in which the increased volatility from the arrival of a large piece of good news and a large piece of bad news creates asymmetric effects on risk premia. It partially offsets the direct positive effect from the arrival of a large piece of good news, while amplifies the direct negative effect from the arrival of a large piece of bad news. As hedge funds frequently invest in illiquid securities, shocks to market volatility have a longer memory in hedge fund returns than in equity markets and hence should be expected to have a large impact on risk premia.

These two mechanisms can also explain the positive coskewness and low cokurtosis in global macro funds. Global macro funds are less subject to “volatility feedback” because their moderate
use of leverage is partially neutralized by their participation in liquid foreign exchange spot and futures markets. As noted by Christie-David and Chaudhry (2001), there is strong evidence of positive skewness and leptokurtosis in futures markets. As global macro funds trade widely across different market and different asset classes, their broad investment mandate may allow them to diversify away those idiosyncratic distribution properties of specific markets (Anson (2002)). In particular, their frequent employment of derivatives can be used as a hedge against extreme events. Kurtosis for a well-diversified global macro portfolio (co-kurtosis) is much lower than individual kurtosis, specifically 10.07 versus 4.34. Thus, global macro funds can act as a “put option” in a global macro portfolio, since adding a global macro fund can reduce the portfolio’s standard deviation and kurtosis and increase its skewness – in much the same way as equity index put options allow an investor to protect against downside moves in the equity market and thereby reduce negative skewness in his portfolio.

Table 3 shows that equity market neutral funds have the lowest average relative kurtosis both for single funds (6.02) and for portfolios of many funds (approaches 3.37 as $n \to \infty$). This is consistent with their neutral exposure to various systematic risks. Table 2 shows that they have moderate negative skewness or coskewness, which is consistent with their holdings of illiquid assets. This undesirable negative skewness can be reduced through the inclusion of derivatives into a single-strategy portfolio of equity market neutral funds.

5 Conclusion

In this paper, we decompose each of the four portfolio moments into a function of portfolio size, thereby separating the effect of diversification on the distribution moments into specific components. We explicitly demonstrate that diversification reduces skewness, improves variance and kurtosis. Covariance, coskewness and cokurtosis, rather than individual hedge fund variance, skewness and kurtosis, matter most in portfolio diversification.
A Procedure to “unsmooth” data

The observed (or smoothed) value $V^*_t$ of a hedge fund at time $t$ can be expressed as a weighted average of the underlying (true) value at time $t$, $V_t$, and the smoothed value at time $t-1$, $V^*_{t-1}$:

$$V^*_t = \alpha V_t + (1 - \alpha)V^*_{t-1}.$$  

Let $B$ be the backshift operator defined by $B^L x_t = x_{t-L}$. Define the following lag function, $L_t(\alpha)$, which is a polynomial $B$, with different coefficients for each of the $t = 1, \ldots, 12$ appraisal cohorts:

$$L_t(\alpha) = \frac{t}{12} + \sum_{L=1}^{\infty} \left[ (1 - \alpha)^{L-1} \left( \frac{12 - t}{12} \right) + (1 - \alpha)^L \left( \frac{t}{12} \right) \right] B^L.$$  

Let $r_t$ and $r^*_t$ denote the true underlying (unobservable) return and the observed return at time $t$ respectively. The monthly smoothed return is given by $r^*_t = \alpha L_m(\alpha)r_t$. We then can derive:

$$r^*_t = \alpha r_t + (1 - \alpha)r^*_{t-1} = \alpha r_t + \alpha(1 - \alpha)r_{t-1} + \alpha(1 - \alpha)^2 r_{t-2} \cdots.$$  

Here we implicitly assume that hedge fund managers use a single exponential smoothing approach. This yields an unsmoothed series with zero first order autocorrelation: $r_t = \alpha^{-1}(r^*_t - (1 - \alpha)r^*_{t-1})$.

Since the stock market indices have around zero autocorrelation coefficients, it seems plausible in the context of the results above to set $1 - \alpha$ equal to the first order autocorrelation coefficient. The newly constructed return series, $r_t$, has the same mean as $r^*_t$, and zero first order autocorrelation (aside from rounding errors), but with higher standard deviation.

B Derivations

The third central moment of a portfolio with $n$ funds can be expressed as

$$E((R_p - E(R_p))^3) = n^{-3} E \left( \sum_{i=1}^{n} (R_i - E(R_i)) \right)^3.$$
Expanding the right hand side further,
\[
= \frac{1}{n^3} \left( E(R_1 - E(R_1))^3 + E(R_2 - E(R_2))^3 + \cdots + E(R_n - E(R_n))^3 \right) \\
+ \frac{3}{n^3} \left( \sum_{i=1}^{n} \sum_{k=1, i \neq k}^{n} E(R_i - E(R_i))^2 E(R_k - E(R_k)) \right) \\
+ \frac{6}{n^3} \left( \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} \sum_{k=1, i \neq j \neq k}^{n} E(R_i - E(R_i)) E(R_j - E(R_j)) E(R_k - E(R_k)) \right).
\]

We can re-write the third central moment in terms of the average comoments for the total population of available funds within the strategy class:
\[
E(R_p - E(R_p))^3 \approx n^{-2} \left[ \text{skew}(i) + 3(n - 1) \text{coskew}(i, i, k) + (n - 1)(n - 2) \text{coskew}(i, j, k) \right].
\]

Similarly, the fourth central moment of a portfolio with \(n\) funds is:
\[
E(R_p - E(R_p))^4 = n^{-4} \left( \sum_{i=1}^{n} (R_i - E(R_i)) \right)^4 \\
= n^{-4} \sum_{i=1}^{n} \left[ \text{kurt}(i) + 4 \sum_{k=1, \neq i}^{n} \text{cokurt}(i, i, i, k) + 6 \sum_{k=1, \neq i, l=1, \neq i, k}^{n} \text{cokurt}(i, i, k, l) \\
+ \sum_{j=1, \neq i, k=1, \neq i, j=1, \neq i, j, k}^{n} \text{cokurt}(i, j, k, l) + 3n \sum_{k=1, \neq i}^{n} \text{cokurt}(i, i, k) \right].
\]

Substituting for the comoment averages based on the total available population of funds, we obtain:
\[
E(R_p - E(R_p))^4 \approx n^{-3} \left[ \text{kurt}(i) + 4(n - 1) \text{cokurt}(i, i, i, k) + 6(n - 1)(n - 2) \text{cokurt}(i, i, k, l) \\
+ (n - 1)(n - 2)(n - 3) \text{cokurt}(i, j, k, l) + 3(n - 1) \text{cokurt}(i, i, k) \right].
\]

To obtain relative skewness and kurtosis, and to convert from variance to standard deviation, we require an additional assumption. Critically, we assume that the portfolio (sample) size is sufficiently large that idiosyncratic risks are diversified away. Consequently, portfolio variance asymptotically approaches a constant – specifically its systematic risk component. This implies:
\[
\lim_{n \to \infty} E(\sigma_p) = \left( E(\sigma_p^2) \right)^{\frac{1}{2}}, \quad \lim_{n \to \infty} E(\sigma_p^3) = \left( E(\sigma_p^2) \right)^{\frac{3}{2}}, \quad \lim_{n \to \infty} E(\sigma_p^4) = \left( E(\sigma_p^2) \right)^2,
\]
\[
\lim_{n \to \infty} E \left( \frac{\text{skew}(i)}{\sigma_p^3} \right) = \frac{E(\text{skew}(i))}{E(\sigma_p^3)^{\frac{3}{2}}}, \quad \lim_{n \to \infty} E \left( \frac{\text{kurt}(i)}{\sigma_p^4} \right) = \frac{E(\text{kurt}(i))}{E(\sigma_p^4)^{\frac{2}{2}}}.
\]

Using these asymptotic results, we obtain our approximate decomposition formulae (2)–(4).
C Bootstrap Procedure

For clarity, we present only the procedure for constructing bootstrap confidence intervals of the standard deviation estimates. Analogous procedures are used for the skewness and kurtosis estimates. Let each hedge fund strategy be denoted by \( s \). We conduct a bootstrap simulation for each “strategy” sample (with \( N_s \) funds). During each bootstrap iteration, \( b = 1, \ldots, B \):

1. We randomly draw \( V \) funds with replacement\(^2\) to form a bootstrap sample, where \( V \geq N_s \);

2. Based on these \( V \) funds, we calculate \( \text{var}_b(i) \) and \( \text{cov}_b(i,j) \) and substitute them into our decomposition formula (2). Using (2), we calculate a series of standard deviation estimates \( (\hat{D}_{bn}) \) corresponding to its value at \( n = 1, \ldots, N_s \).

3. We sequentially select \( n = 1, \ldots, N_s \) funds from the bootstrap sample to form a portfolio. For each portfolio of \( n \) funds, we calculate the unbiased estimate of standard deviation \( (\hat{D}_{bn}) \) using (6).

Upon completion of the simulation, we calculate \( \bar{\hat{D}}_n = \frac{1}{B} \sum_{b=1}^{B} \hat{D}_{bn} \) and \( \bar{D}_n = \frac{1}{B} \sum_{b=1}^{B} D_{bn} \). For each sample size \( n \), we sort the moment statistics series \( \{D_{bn}\} \) in ascending order. Then, the \((1 - \alpha)\) two-sided confidence interval for sample size \( n \) is given by: \( D_0 \in [\hat{D}_{b(\alpha/2),n}, \hat{D}_{b(1-\alpha/2),n}] \) where \( D_0 \) represents the true, unobservable standard deviation. Our main interest is whether or not our average estimate obtained from our decomposition approximation (i.e. \( \bar{\hat{D}}_n \)) lies within this confidence interval.

\(^2\)As a robustness check, we also construct bootstrap confidence intervals based on draws without replacement. This alternative method produced similar results.
D References


Table 1: **Statistical summary of reported and “unsmoothed” hedge fund returns.** Reported values are calculated from monthly net-of-all-fee returns and averaged across funds. Kurtosis represents the excess kurtosis. First-order to fourth-order autocorrelation is given by AC(1)–AC(4).

### Based on reported returns

<table>
<thead>
<tr>
<th>Fund Type</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>AC(1)</th>
<th>AC(2)</th>
<th>AC(3)</th>
<th>AC(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arbitrage</td>
<td>0.96</td>
<td>3.01</td>
<td>-1.14</td>
<td>5.93</td>
<td>0.30</td>
<td>0.15</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>Dedicated Short Bias</td>
<td>-0.18</td>
<td>7.92</td>
<td>0.18</td>
<td>3.37</td>
<td>0.14</td>
<td>0.02</td>
<td>0.03</td>
<td>-0.04</td>
</tr>
<tr>
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<td>0.89</td>
<td>2.37</td>
<td>-0.78</td>
<td>6.36</td>
<td>0.25</td>
<td>0.08</td>
<td>-0.04</td>
<td>0.02</td>
</tr>
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<td>Equity Market Neutral</td>
<td>0.54</td>
<td>2.70</td>
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<td>2.82</td>
<td>0.20</td>
<td>0.03</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Fund of Funds</td>
<td>0.75</td>
<td>3.34</td>
<td>-0.16</td>
<td>3.83</td>
<td>0.17</td>
<td>0.04</td>
<td>-0.00</td>
<td>-0.04</td>
</tr>
<tr>
<td>Global Macro</td>
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<td>5.23</td>
<td>1.06</td>
<td>7.63</td>
<td>0.11</td>
<td>0.01</td>
<td>-0.00</td>
<td>-0.03</td>
</tr>
<tr>
<td>Long/Short Equity</td>
<td>1.34</td>
<td>5.83</td>
<td>0.00</td>
<td>3.35</td>
<td>0.09</td>
<td>-0.00</td>
<td>0.01</td>
<td>-0.03</td>
</tr>
<tr>
<td>Merger Arbitrage</td>
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<td>4.96</td>
<td>0.10</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.03</td>
</tr>
<tr>
<td>Emerging Markets</td>
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<td>7.85</td>
<td>-0.86</td>
<td>5.79</td>
<td>0.10</td>
<td>-0.01</td>
<td>-0.00</td>
<td>-0.02</td>
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<tr>
<td>All Fund Portfolio</td>
<td>1.00</td>
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<td>4.62</td>
<td>0.12</td>
<td>0.00</td>
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<td>-0.02</td>
</tr>
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</table>

### Based on “unsmoothed” returns

<table>
<thead>
<tr>
<th>Fund Type</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>AC(1)</th>
<th>AC(2)</th>
<th>AC(3)</th>
<th>AC(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arbitrage</td>
<td>0.96</td>
<td>3.99</td>
<td>-0.91</td>
<td>5.46</td>
<td>0.00</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>Dedicated Short Bias</td>
<td>-0.26</td>
<td>7.62</td>
<td>0.22</td>
<td>3.22</td>
<td>0.00</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>Distressed Securities</td>
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<td>-0.02</td>
</tr>
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<td>0.55</td>
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<td>2.94</td>
<td>0.01</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
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<td>-0.02</td>
<td>-0.01</td>
<td>-0.03</td>
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<td>1.03</td>
<td>7.16</td>
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<td>-0.01</td>
<td>-0.03</td>
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<tr>
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<td>6.35</td>
<td>0.01</td>
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<td>-0.03</td>
<td>-0.00</td>
<td>-0.03</td>
</tr>
<tr>
<td>Merger Arbitrage</td>
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<td>2.06</td>
<td>-0.46</td>
<td>4.65</td>
<td>0.00</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>0.23</td>
<td>9.63</td>
<td>-0.91</td>
<td>5.91</td>
<td>0.01</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>All Fund Portfolio</td>
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<td>5.94</td>
<td>-0.13</td>
<td>4.55</td>
<td>0.00</td>
<td>-0.03</td>
<td>-0.01</td>
<td>-0.02</td>
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</table>
Table 2: Expected standard deviation, expected skewness and the corresponding coskewness decomposition for single-strategy fund of hedge fund portfolios. Reported values are calibrated and calculated from monthly net-of-all-fee total returns. Results are based on bootstrap simulation averages of portfolios of size \( n \) randomly drawn from all available funds within its strategy class. Expected standard deviations in columns 2 and 3 are based on \( E(\sigma(n)) \approx n^{-1/2}[\text{var}(i) + (n - 1)\text{cov}(i, j)]^{1/2} \) evaluated at \( n = 1 \) and \( n \to \infty \), respectively. Values in columns 4–6 correspond to the comoments in our skewness decomposition formula (3). The values in columns 7 and 8 correspond to the skewness decomposition formula evaluated at \( n = 1 \) and \( n \to \infty \), respectively.

<table>
<thead>
<tr>
<th>Portfolio Type</th>
<th>Expected Std. Dev.</th>
<th>Coskew ( (\cdot) ) Decomposition</th>
<th>Expected Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( n = 1 )</td>
<td>( n \to \infty )</td>
<td>( (i, i, i) )</td>
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<tr>
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<td>-68.47</td>
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<td>94.39</td>
</tr>
<tr>
<td>Distressed Securities</td>
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<td>-19.07</td>
</tr>
<tr>
<td>Equity Market Neutral</td>
<td>3.06</td>
<td>0.99</td>
<td>-11.09</td>
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<tr>
<td>Fund of Funds</td>
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<td>-13.41</td>
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<tr>
<td>Global Macro</td>
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<td>157.29</td>
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<td>Long/Short Equity</td>
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<td>Merger Arbitrage</td>
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<td>1.20</td>
<td>-4.01</td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>9.68</td>
<td>6.99</td>
<td>-825.44</td>
</tr>
<tr>
<td>All Fund Portfolio</td>
<td>6.06</td>
<td>2.56</td>
<td>-26.32</td>
</tr>
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</table>
Table 3: **Expected kurtosis and corresponding cokurtosis decomposition for single-strategy hedge fund portfolios.** Reported values are calibrated and calculated from monthly net-of-all-fee total returns. Results are based on bootstrap simulation averages of portfolios of size $n$ randomly drawn from all available funds within its strategy class. Values in columns 2–5 correspond to the comoments in our kurtosis decomposition formula (4). The values in columns 6 and 7 correspond to the kurtosis decomposition formula evaluated at $n = 1$ and $n \to \infty$, respectively.

<table>
<thead>
<tr>
<th>Hedge Fund Portfolio</th>
<th>Cokurtosis Decomposition ($i, i, i, i$)</th>
<th>($i, i, k, l$)</th>
<th>($i, j, k, l$)</th>
<th>($i, i, j, j$)</th>
<th>Expected Kurtosis $n = 1$</th>
<th>$n \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>129</td>
<td>5,633</td>
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<td>6.46</td>
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<tr>
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<td>9,817</td>
<td>14,136</td>
<td>154,282</td>
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<td>18</td>
<td>3</td>
<td>491</td>
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<td>10,010</td>
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<td>4.34</td>
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<tr>
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<td>475</td>
<td>19,337</td>
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<td>Merger Arbitrage</td>
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<td>40</td>
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<td>Emerging Markets</td>
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<td>274,319</td>
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<tr>
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<td>605</td>
<td>286</td>
<td>14,838</td>
<td>7.78</td>
<td>6.71</td>
</tr>
</tbody>
</table>
Figure 1: **Expected standard deviation as a function of the number of funds ($n$) in a portfolio of convertible arbitrage funds.** The figure shows the upper and lower bounds for the 99% and 90% bootstrap confidence intervals based on the unbiased estimates of standard deviation of the $n$ funds randomly drawn in the portfolio. Plotted are the average value of standard deviation based on the decomposition in formula (2) and the average of unbiased estimates of standard deviation based on the actual returns of the randomly drawn portfolio of funds.
Figure 2: Expected skewness as a function of the number of funds ($n$) in a portfolio of equity market neutral funds. The figure shows the upper and lower bounds for the 99% and 90% bootstrap confidence intervals based on the unbiased estimates of standard deviation of the $n$ funds randomly drawn in the portfolio. Plotted are the average value of skewness based on the decomposition in formula (3) and the average of unbiased estimates of skewness based on the actual returns of the randomly drawn portfolio of funds.
Figure 3: Expected kurtosis as a function of the number of funds ($n$) in a portfolio of distressed securities funds. The figure shows the upper and lower bounds for the 99% and 90% bootstrap confidence intervals based on the unbiased estimates of standard deviation of the $n$ funds randomly drawn in the portfolio. Plotted are the average value of kurtosis based on the decomposition in formula (4) and the average of unbiased estimates of kurtosis based on the actual returns of the randomly drawn portfolio of funds.
Figure 4: The contribution of the skewness decomposition comoments. The three component weights of our decomposition formula (3) are plotted against the number of funds in the portfolio \(n\). Curve A is the value of \(n^{-2}\) (the weight of the skew(i) term). Curve B is the value of \(3n^{-2}(n - 1)\) (the weight of the coskew(i,i,k) term). Curve C is the value of \(n^{-2}(n - 1)(n - 2)\) (the weight of the coskew(i,j,k) term).
Figure 5: **The contribution of the kurtosis decomposition comoments.** The five component weights of our decomposition formula (4) are plotted against the number of funds in the portfolio \((n)\). Curve A is the value of \(n^{-3}\) (the weight of the \(\overline{\text{kurt}}(i)\) term). Curve B is the value of \(4n^{-3}(n - 1)\) (the weight of the \(\text{cokurt}(i, i, i, k)\) term). Curve C is the value of \(6n^{-3}(n - 1)(n - 2)\) (the weight of the \(\text{cokurt}(i, i, k, l)\) term). Curve D is the value of \(n^{-3}(n - 1)(n - 2)(n - 3)\) (the weight of the \(\text{cokurt}(i, j, k, l)\) term). Curve E is the value of \(3n^{-3}(n - 1)\) (the weight of the \(\text{cokurt}(i, i, k, k)\) term).