Determining and Forecasting High-Frequency Value-at-Risk by Using Lévy Processes

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Stylized Facts of High-Frequency Stock Market Data

- Random durations (Dacorogna et al. (2001))
- Distributional properties
  - ▷ Fatter tails in the unconditional return distributions.
    (Bollerslev et al. (1992), Marinelli et al. (2000))
  - ▷ Stock returns are not independently and identically distributed.
    (Sun et al. (2006a))
- Autocorrelation (Bollwerslev et al. (2000), Wood et al. (1985))
- Seasonality (Gourieroux and Jasiak (2001))
- Clustering
  - ▷ Volatility clustering. (Engle (2000))
  - ▷ Trade duration clustering (Sun et al. (2006b))
- Long-range dependence. (Sun et al. (2006a))

Motivation

Modeling Irregularity and Roughness of Price Movement

- Capturing the stylized facts observed in high-frequency data
- Establishing a model for the study of price dynamics
- Simulating price movement based on the established model
- Testing the goodness of fit for the established model

Modeling Dependence Structure

- Capturing dynamic continuity of the underlying asset

Model Application

- Computing Value-at-Risk

Lévy Processes with Specification

- Lévy processes have become increasingly popular in mathematical finance because they can describe the observed behavior of financial markets in a more accurate way. They capture jumps, heavy-tails, and skewness observed in the market for asset price processes. Moreover, Lévy processes provide the appropriate option pricing framework to model implied volatilities across strike prices and across maturities with respect to the "risk-neutral" assumption.

- We review the definition of Lévy processes as well as one specific form (the Lévy fractional stable motion) and two extensions of infinitely divisible distributions (the Lévy stable distribution and fractional Brownian motion) that we use in this paper. Further details can be found in Sato (1999). Lévy fractional stable motion and fractional Brownian motion are often referred to as fractal processes.

Why Fractal Processes?

- "The reasons are that the main feature of price records is roughness and that the proper language of the theory of roughness in nature and culture is fractal geometry" (Mandelbrot (2005)).
  - Mandelbrot (1982): The fractal geometry of nature. Freeman, New York.
  - Mandelbrot (1997): Fractals and scaling in finance. Springer, New York.
  - Mandelbrot (2002): Gaussian self-affinity and fractals. Springer, New York.

- Custom has made the increments' ratio be viewed as "normal" and thought the highly anomalous ratio has the limit $H = 1/2$.

- Being the same at all instants in all financial data is a very important property. It has simplicity. But it also has a big flaw – a limit equal to $1/2$ is not available as parameter to be fitted to the data.

- The fractal processes allow $H \neq 1/2$.

- A key feature of fractal processes is that it measures roughness by α and the value and/or the distribution of α is directly observable.

- The speed of volatility variation can be considered by the fractal processes.

What are the Fractal Processes?

- Fractal processes (self-similar processes) are invariant in distribution with respect to changes of time and space scale. The scaling coefficient or self-similarity index is a non-negative number denoted by H, the Hurst parameter.

- Lamperti (1962) first introduced semi-stable processes (which we nowadays call self-similar processes).

- If $\{X(t+h) - X(h), t \in T\} \stackrel{d}{=} \{X(t) - X(0), t \in T\}$ for all $h \in T$, the real-valued process $\{X(t), t \in T\}$ has stationary increments. Samorodnisky and Taqqu (1994) provide a succinct expression of self-similarity: $\{X(at), t \in T\} \stackrel{d}{=} \{a^H X(t), t \in T\}$. The process $\{X(t), t \in T\}$ is called H-sssi if it is self-similar with index H and has stationary increments.

- In our study, two fractal processes are employed:
  - ▷ fractional Gaussian noise
  - ▷ Lévy fractional stable noise

Fractional Gaussian noise

- For a given $H \in (0,1)$ there is basically a single Gaussian H-sssi process, namely fractional Brownian motion (fBm) that was first introduced by Kolmogorov (1940). Mandelbrot and Wallis (1968) and Taqqu (2003) clarify the definition of fBm as a Gaussian H-sssi process $\{B_H(t)\}_{t\in R}$ with $0 < H < 1$.
- The fractional Brownian motion (fBm) has the integral representation

$$B_H(t) \quad = \quad \int_{-\infty}^{\infty}\left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right)B(du)$$

  where $x_+ := \max(x, 0)$ and $B(du)$ represents a symmetric Gaussian independently scattered random measure.
- As to the fractional Brownian motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional Gaussian noise (fGn), which is, for $j = 0, \pm 1, \pm 2, \ldots$, $Y_j = B_H(j+1) - B_H(j)$.
- The main difference between fractional Brownian motion and ordinary Brownian motion is that the increments in Brownian motion are independent while in fractional Brownian motion they are dependent.

Lévy fractional stable noise

- The most commonly used extension of fBm to the α-stable case is the fractional Lévy stable motion, which is defined by following integral representation

$$Z_\alpha^H(t) \quad = \quad \int_{-\infty}^{\infty} \Big((t-u)_+^{H-\frac{1}{\alpha}} - (-u)_+^{H-\frac{1}{\alpha}} \Big) Z_\alpha(du)$$

  where Z_α is a symmetric α-stable independently scattered random measure.

- As to the fractional stable motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional stable noise (fsn), which is, for $j = 0, \pm 1, \pm 2, \ldots$, $Y_j = Z_\alpha^H(j+1) - Z_\alpha^H(j)$.

Lévy stable Distribution

- Stable distribution requires four parameters for complete description:
  - ▷ an index of stability $\alpha \in (0, 2]$ (also called the tail index),
  - ▷ a skewness parameter $\beta \in [-1, 1]$,
  - ▷ a scale parameter $\gamma > 0$,
  - ▷ a location parameter $\zeta \in \Re$.

- There is unfortunately no closed-form expression for the density function and distribution function of a stable distribution. Lévy (1937) gives the definition of the stable distribution: A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma \geq 0$ and real ζ such that its characteristic function has the following form:

$$E\exp(i\theta X) = \begin{cases} \exp\{-\gamma^{\alpha}|\theta|^{\alpha}(1 - i\beta(\mathrm{sign}\theta)\tan\frac{\pi\alpha}{2}) + i\zeta\theta\} & if \quad \alpha \neq 1 \\ \exp\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\mathrm{sign}\theta)\ln|\theta|) + i\zeta\theta\} & if \quad \alpha = 1 \end{cases}$$

  and,

$$\mathrm{sign}\ \theta = \begin{cases} 1 & if \quad \theta > 0 \\ 0 & if \quad \theta = 0 \\ -1 & if \quad \theta < 0 \end{cases}$$

- Mandelbrot (1997) and Rachev and Mittnik (2000) apply it in finance.

The model

- The parametric approach for VaR estimation is based on the assumption that the financial returns R_t are a function of two components μ_t and ε_t (i.e., $R_t = f(\mu_t, \varepsilon_t)$). R_t can be regarded as a function of ε_t conditional on a given μ_t; typically this function takes a simple linear form $R_t = \mu_t + \varepsilon_t = \mu_t + \sigma_t u_t$. Usually μ_t is referred to as the location component and σ_t the scale component. u_t is an independent and identically distributed (i.i.d.) random variable that follows a probability density function f_u. VaR based on information up to time t is then

$$V\tilde{a}R_t := q_\alpha(R_t) = -\tilde{\mu}_t - \tilde{\sigma}\, q_\alpha(u)$$

where $q_\alpha(u)$ is the α-quantile implied by f_u.

Computing Value-at-Risk

- Unconditional parametric approaches set μ_t and σ_t as constants, therefore the returns R_t are i.i.d random variables with density $\sigma^{-1} f_u(\sigma^{-1}(R_t - \mu))$. Conditional parametric approaches set location component and scale component as functions not constants.

- The typical time-varying conditional location setting is the ARMA(r, m) processes. That is, the conditional mean equation is:

$$\mu_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i R_{t-i} + \sum_{j=1}^{m} \beta_j \varepsilon_{t-j}$$

- The typical time-varying conditional variance setting is GARCH(p, q) processes given by

$$\sigma_t^2 = \kappa + \sum_{i=1}^{\mathrm{p}} \gamma_i \sigma_{t-i}^2 + \sum_{j=1}^{\mathrm{q}} \theta_j \varepsilon_{t-j}^2$$

- Different distributional assumptions for the innovation distribution f_u can be made. In the empirical analysis below, distributional assumptions analyzed for the parametric approaches are the normal distribution, fractional Gaussian noise, fractional Lévy stable noise, and Lévy stable distribution.

The Goodness of Fit Tests

- Kolmogorov-Simirnov distance (KS)

$$KS = \sup_{x \in \Re} \left| F_s(x) - \tilde{F}(x) \right|,$$

- Anderson-Darling distance (AD)

$$AD = \sup_{x \in \Re} \frac{\left| F_s(x) - \tilde{F}(x) \right|}{\sqrt{\tilde{F}(x)(1 - \tilde{F}(x))}},$$

- Cramer Von Mises distance (CVM)

$$CVM = \int_{-\infty}^{\infty} \left(F_s(x) - \tilde{F}(x) \right)^2 d\tilde{F}(x),$$

- Kuiper distance (K)

$$K = \sup_{x \in \Re} \left(F_s(x) - \tilde{F}(x) \right) + \sup_{x \in \Re} \left(\tilde{F}(x) - F_s(x) \right).$$

The Data

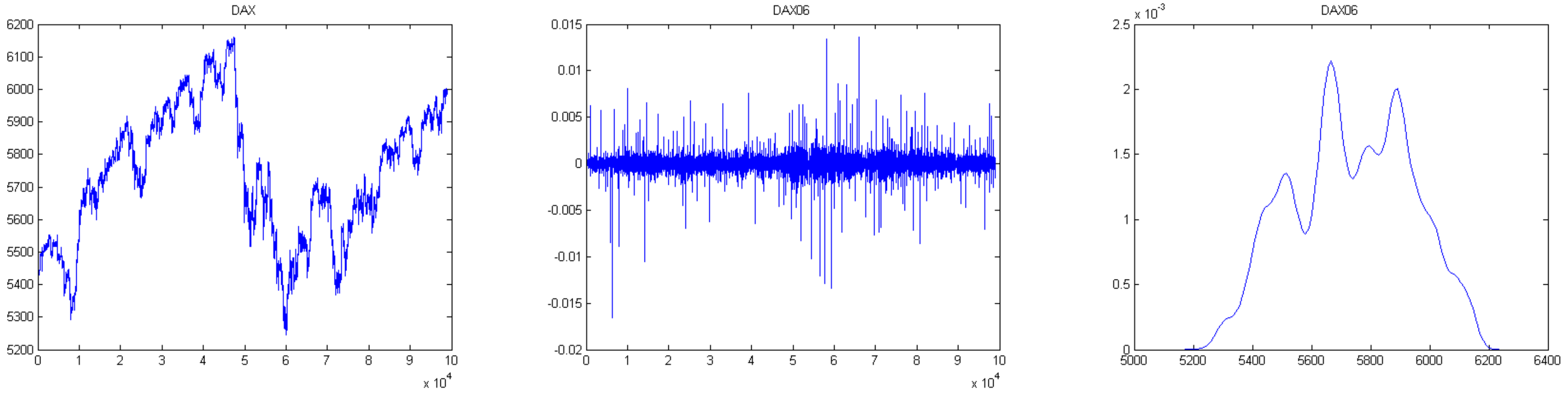
- In our study, we consider the Deutsche Aktien Xchange (DAX) index from January 2 to September 30, 2006 that were aggregated to the 1-minute frequency level.

- The DAX index is a stock market index whose components include 30 blue chip German stocks that are traded on the Frankfurt Stock Exchange. Starting in 2006, the DAX index is calculated every second. In our original dataset, the DAX index is sampled at the one-second level.

- Employing high-frequency data has several advantages compared to the low-frequency data. First, with a very large amount of observations, high-frequency data offers a higher level of statistical significance. Second, high-frequency data are gathered at a low level of aggregation, thereby capturing the heterogeneity of players in financial markets. Third, day-trading strategies require the analysis of high-frequency data.

The data

- Data plot.



Computing Value-at-Risk

The method

- In the first experiment,we calculate **95%**-VaR values and **99%**-VaR values for the entire data sample.
- We compute the in-sample 95% VaR and 99% VaR with a horizon of six months. Our dataset contains nine months of data. In order to ensure randomness of the data from which VaR is computed, after each computation, we shift the next starting point of the training dataset for VaR computation two weeks afterwards.

Computing Value-at-Risk

The method

- In the second experiment, we split the dataset into two subsets: an in-sample (training) set and an out-of-sample (forecasting) set. The purpose of this second experiment is mainly to check the prediction power of parametric VaR values computed from the in-sample set.

- In order to test the predictive power of the VaR value computed for each model, in our second experiment we test the following six predictions using different size of training and prediction datasets:
  - ▷ Training period is 6 months
    and forecast for 6 months, 3 months, 1 month, 1 week, 1 day and 1 hour;
  - ▷ Training period is 3 months
    and forecast for 3 months, 1 month, 1 week, 1 day and 1 hour;
  - ▷ Training period is 1 month
    and forecast for 1 month, 1 week, 1 day and 1 hour;
  - ▷ Training period is 1 week
    and forecast for 1 week, 1 day and 1 hour;
  - ▷ Training period is 1 day
    and forecast for 1 day and 1 hour;
  - ▷ Training period is 1 hour
    and forecast for 1 hour.

In-sample goodness of fit

Table 1: Summary of in-sample goodness of fit statistics for different models.

a. AD-statistic	AD_{mean}	AD_{std}	AD_{median}	AD_{max}	AD_{min}	AD_{range}
ARMA-GARCH-fGn	46.6768	54.3660	13.7335	55.8541	21.6282	34.2259
ARMA-GARCH-fsn	44.1625	53.2522	15.2382	64.6001	1.3917	63.2084
ARMA-GARCH-nor	46.7177	54.3751	13.7480	58.5747	21.0690	37.5057
ARMA-GARCH-sta	45.4108	53.5900	14.3638	95.2204	2.9886	92.2318
ARMA-GARCH-gev	46.6401	54.2656	13.7441	60.2271	21.0914	39.1357
ARMA-GARCH-gpd	51.2203	54.4755	20.2070	109.5363	3.5018	106.0244
b. KS-statistic	KS_{mean}	KS_{std}	KS_{median}	KS_{max}	KS_{min}	KS_{range}
ARMA-GARCH-fGn	0.4998	0.4992	0.0034	0.5285	0.4887	0.0398
ARMA-GARCH-fsn	0.4938	0.4965	0.0261	0.9725	0.2745	0.6980
ARMA-GARCH-nor	0.5003	0.4994	0.0043	0.5455	0.4893	0.0562
ARMA-GARCH-sta	0.5089	0.4974	0.0622	0.9725	0.3910	0.5815
ARMA-GARCH-gev	0.5000	0.4989	0.0057	0.5775	0.4825	0.0950
ARMA-GARCH-gpd	0.5698	0.5198	0.1335	1.0000	0.4165	0.5835
c. CVM-statistic	CVM_{mean}	CVM_{std}	CVM_{median}	CVM_{max}	CVM_{min}	CVM_{range}
ARMA-GARCH-fGn	449.5326	517.0503	203.7237	896.6917	82.9310	813.7607
ARMA-GARCH-fsn	445.2936	515.2956	202.2463	1473.6038	34.4169	1439.1868
ARMA-GARCH-nor	449.6701	517.3712	203.7739	889.0445	82.9532	806.0913
ARMA-GARCH-sta	454.7954	516.7259	230.4694	2985.6218	57.5066	2928.1152
ARMA-GARCH-gev	449.2438	517.0805	203.8510	886.1477	83.0288	803.1189
ARMA-GARCH-gpd	524.3399	521.2781	370.3320	1978.9581	52.5637	1926.3945
d. Kuiper-statistic	$Kuiper_{mean}$	$Kuiper_{std}$	$Kuiper_{median}$	$Kuiper_{max}$	$Kuiper_{min}$	$Kuiper_{range}$
ARMA-GARCH-fGn	0.9931	0.9937	0.0029	0.9985	0.9757	0.0227
ARMA-GARCH-fsn	0.9693	0.9862	0.0473	0.9985	0.5125	0.4860
ARMA-GARCH-nor	0.9931	0.9938	0.0029	0.9990	0.9750	0.0240
ARMA-GARCH-sta	0.9796	0.9877	0.0224	0.9990	0.6550	0.3440
ARMA-GARCH-gev	0.9913	0.9925	0.0048	0.9990	0.9570	0.0420
ARMA-GARCH-gpd	0.9696	0.9773	0.0287	1.0000	0.6505	0.3495

Out-of-sample goodness of fit

Table 2: Goodness of fit statistics for out-of-sample one week forecasting of different models.

a. AD-statistic	AD_{mean}	AD_{std}	AD_{median}	AD_{max}	AD_{min}	AD_{range}
ARMA-GARCH-fGn	30.1821	22.5228	13.6283	55.2241	21.5834	33.6407
ARMA-GARCH-fsn	27.6038	22.4110	12.2880	68.1149	1.2129	66.9021
ARMA-GARCH-nor	30.1927	22.5228	13.6421	59.2046	21.1361	38.0684
ARMA-GARCH-sta	28.8034	22.3886	13.0021	101.7023	2.6941	99.0082
ARMA-GARCH-gev	30.1205	22.5005	13.5541	59.8893	20.7111	39.1782
ARMA-GARCH-gpd	32.3273	23.9319	15.3931	108.9876	4.1084	104.8792
b. KS-statistic	KS_{mean}	KS_{std}	KS_{median}	KS_{max}	KS_{min}	KS_{range}
ARMA-GARCH-fGn	0.5018	0.5006	0.0049	0.5375	0.4905	0.0470
ARMA-GARCH-fsn	0.4965	0.4985	0.0278	0.9555	0.2820	0.6734
ARMA-GARCH-nor	0.5020	0.5010	0.0055	0.5615	0.4880	0.0735
ARMA-GARCH-sta	0.5105	0.4990	0.0617	0.9653	0.4049	0.5603
ARMA-GARCH-gev	0.5018	0.5005	0.0064	0.5705	0.4846	0.0858
ARMA-GARCH-gpd	0.5700	0.5210	0.1333	1.0000	0.4049	0.5951
c. CVM-statistic	CVM_{mean}	CVM_{std}	CVM_{median}	CVM_{max}	CVM_{min}	CVM_{range}
ARMA-GARCH-fGn	226.5630	92.4072	245.5856	950.0136	82.5743	867.4392
ARMA-GARCH-fsn	223.6907	91.6806	244.8920	1873.2304	34.0265	1839.2039
ARMA-GARCH-nor	226.6043	92.4969	245.5955	948.5973	82.7246	865.8726
ARMA-GARCH-sta	229.7204	92.0018	264.0086	2852.7912	77.7136	2775.0774
ARMA-GARCH-gev	225.5252	92.2819	243.0575	933.6036	82.4613	851.1423
ARMA-GARCH-gpd	248.8990	94.0770	282.0017	1929.6210	55.7156	1873.9054
d. Kuiper-statistic	$Kuiper_{mean}$	$Kuiper_{std}$	$Kuiper_{median}$	$Kuiper_{max}$	$Kuiper_{min}$	$Kuiper_{range}$
ARMA-GARCH-fGn	0.9935	0.9940	0.0032	1.0000	0.9715	0.0285
ARMA-GARCH-fsn	0.9698	0.9870	0.0481	0.9990	0.5362	0.4627
ARMA-GARCH-nor	0.9935	0.9940	0.0032	1.0000	0.9725	0.0275
ARMA-GARCH-sta	0.9801	0.9885	0.0231	0.9995	0.6876	0.3118
ARMA-GARCH-gev	0.9918	0.9930	0.0052	0.9995	0.9592	0.0402
ARMA-GARCH-gpd	0.9703	0.9780	0.0299	1.0000	0.6425	0.3575

Empirical Results

- VaR values calculated by Kernel estimator (empirical) and ARMA(1,1)-GARCH(1,1) with different residuals (i.e., normal, stable, fractional stable noise, and fractional Gaussian noise).

	95%					99%				
	Empirical	Normal	stable	FSN	FGN	Empirical	Normal	stable	FSN	FGN
No. 1	1.4018	1.8682	1.3874	1.3929	1.8727	2.6607	2.6382	3.7350	3.6972	2.6419
No. 2	1.5218	2.0601	1.5090	1.4778	2.0544	2.8501	2.9267	3.9461	3.8371	2.9178
No. 3	1.5402	2.3234	1.5562	1.6189	2.3158	2.9148	3.2939	4.3943	4.0179	3.2580
No. 4	1.4901	1.8864	1.3882	1.4224	1.9109	2.8036	2.6445	4.4273	3.5853	2.6159
No. 5	1.3861	1.7735	1.5159	1.4412	1.8549	2.4768	2.6127	4.5385	4.4081	2.8099
No. 6	1.6819	1.5163	1.5085	1.7600	1.9079	2.7741	2.6503	5.6604	6.4754	2.9549

- Difference between VaR values calculated by Kernel estimator (empirical) and ARMA(1,1)-GARCH(1,1) with different residuals (i.e., normal, stable, fractional stable noise, and fractional Gaussian noise).

	95%				99%			
	Normal	stable	FSN	FGN	Normal	stable	FSN	FGN
6 M	-0.4044	0.0144	0.0089	-0.4709	0.0225	-1.0743	-1.0365	0.0188
3 M	-0.5383	0.0128	0.0440	-0.5326	-0.0766	-1.0960	-0.9870	-0.0677
1 M	-0.7832	-0.0160	-0.0787	-0.7756	-0.3791	-1.4795	-1.1031	-0.3432
1 W	-0.3963	0.1019	0.0677	-0.4208	0.1591	-1.6237	-0.7817	0.1877
1 D	-0.3874	-0.1298	-0.0551	-0.4688	-0.1359	-2.0617	-1.9313	-0.3331
1 H	0.1656	0.1734	-0.0781	-0.2260	0.1238	-2.8863	-3.7013	-0.1808
$\lvert \sum \rvert$	2.6752	0.4483	0.3325	2.8947	0.8970	10.2215	9.5409	1.1313

Backtesting (Kupiec Test) for $VaR_{\alpha=0.05}$.

	6 months	(236160)	3 months	(124800)	1 month	(40320)	1 week	(9600)	1 day	(1920)	1 hour	(240)
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour	13	25	12	25	4	11	13	21	8	16	8	6
(240)	*0.0542*	*0.1042*	*0.0500*	*0.1042*	*0.0167*	*0.0458*	*0.0542*	*0.0875*	*0.0333*	*0.0667*	*0.0333*	*0.0250*
1 day	35	71	49	105	33	73	47	110	59	95		
(1920)	*0.0182*	*0.0370*	*0.0255*	*0.0547*	*0.0172*	*0.0380*	*0.0245*	*0.0573*	*0.0307*	*0.0495*		
1 week	117	246	185	425	187	458	260	545				
(9600)	*0.0122*	*0.0256*	*0.0193*	*0.0443*	*0.0195*	*0.0477*	*0.0271*	*0.0568*				
1 month	545	1124	686	1602	661	1624						
(40320)	*0.0135*	*0.0279*	*0.0170*	*0.0397*	*0.0164*	*0.0403*						
3 months	1572	3165	1810	4222								
(124800)	*0.0126*	*0.0254*	*0.0145*	*0.0338*								
6 months	2744	5538										
(236160)	*0.0116*	*0.0235*										
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour	25	14	25	14	13	4	18	13	15	8	6	6
(240)	*0.1042*	*0.0583*	*0.1042*	*0.0583*	*0.0542*	*0.0167*	*0.0750*	*0.0542*	*0.0625*	*0.0333*	*0.0250*	*0.0250*
1 day	71	37	98	50	90	33	96	47	93	56		
(1920)	*0.0370*	*0.0193*	*0.0510*	*0.0260*	*0.0469*	*0.0172*	*0.0500*	*0.0245*	*0.0484*	*0.0292*		
1 week	244	113	408	182	542	180	484	260				
(9600)	*0.0254*	*0.0118*	*0.0425*	*0.0190*	*0.0565*	*0.0188*	*0.0504*	*0.0271*				
1 month	1146	556	1608	684	1955	647						
(40320)	*0.0284*	*0.0138*	*0.0399*	*0.0170*	*0.0485*	*0.0160*						
3 months	3259	1588	4221	1801								
(124800)	*0.0261*	*0.0127*	*0.0338*	*0.0144*								
6 months	5728	2781										
(236160)	*0.0243*	*0.0118*										

Empirical Results

Backtesting (Kupiec Test) for $VaR_{\alpha=0.01}$.

	6 months	(236160)	3 months	(124800)	1 month	(40320)	1 week	(9600)	1 day	(1920)	1 hour	(240)
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour	2	0	5	2	1	1	9	2	4	1	3	0
(240)	*0.0083*	*0.0000*	*0.0208*	*0.0083*	*0.0042*	*0.0042*	*0.0375*	*0.0083*	*0.0167*	*0.0042*	*0.0125*	*0.0000*
1 day	13	4	17	10	12	7	24	5	22	4		
(1920)	*0.0068*	*0.0021*	*0.0089*	*0.0052*	*0.0063*	*0.0036*	*0.0125*	*0.0026*	*0.0115*	*0.0021*		
1 week	40	10	62	25	69	36	114	30				
(9600)	*0.0042*	*0.0010*	*0.0065*	*0.0026*	*0.0072*	*0.0038*	*0.0119*	*0.0031*				
1 month	198	48	241	93	257	120						
(40320)	*0.0049*	*0.0012*	*0.0060*	*0.0023*	*0.0064*	*0.0030*						
3 months	593	188	637	268								
(124800)	*0.0048*	*0.0015*	*0.0051*	*0.0021*								
6 months	1055	354										
(236160)	*0.0045*	*0.0015*										
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour	0	2	2	5	1	1	1	9	1	2	0	2
(240)	*0.0000*	*0.0089*	*0.0083*	*0.0208*	*0.0042*	*0.0042*	*0.0042*	*0.0375*	*0.0042*	*0.0083*	*0.0000*	*0.0083*
1 day	4	13	11	18	6	11	2	25	4	20		
(1920)	*0.0021*	*0.0068*	*0.0057*	*0.0094*	*0.0031*	*0.0057*	*0.0010*	*0.0130*	*0.0021*	*0.0104*		
1 week	11	40	24	65	35	69	21	112				
(9600)	*0.0011*	*0.0042*	*0.0025*	*0.0068*	*0.0036*	*0.0072*	*0.0022*	*0.0117*				
1 month	58	197	90	241	133	248						
(40320)	*0.0014*	*0.0049*	*0.0022*	*0.0060*	*0.0028*	*0.0062*						
3 months	213	591	260	651								
(124800)	*0.0017*	*0.0047*	*0.0021*	*0.0052*								
6 months	388	1044										
(236160)	*0.0016*	*0.0044*										

Empirical Results

Admissible VaR violations and violation frequencies.

		$T =$	violation $\alpha = 0.05$	violation frequency $\alpha = 0.05$	violation $\alpha = 0.01$	violation frequency $\alpha = 0.01$
95% VaR	1 hour	240	[6,18]	[0.0250,0.0750]	[4,20]	[0.0166,0.0833]
	1 day	1920	[80,120]	[0.0417,0.0583]	[74,118]	[0.0385,0.0614]
	1 week	9600	[445,515]	[0.0463,0.0536]	[430,530]	[0.0447,0.0552]
	1 month	40320	[1944,2088]	[0.0482,0.0517]	[1915,2118]	[0.0474,0.0525]
	3 months	124800	[6114,6366]	[0.0489,0.0510]	[6061,6419]	[0.0485,0.0514]
	6 months	236160	[11634,11982]	[0.0492,0.0507]	[11562,12054]	[0.0489,0.0510]
99%VaR	1 hour	240	[0,5]	[0.0000,0.0208]	[0,6]	[0.0000,0.0250]
	1 day	1920	[12,26]	[0.0062,0.0135]	[9,29]	[0.0046,0.0151]
	1 week	9600	[80,112]	[0.0083,0.0117]	[73,119]	[0.0076,0.0123]
	1 month	40320	[370,436]	[0.0092,0.0108]	[357,450]	[0.0088,0.0111]
	3 months	124800	[1190,1306]	[0.0095,0.0105]	[1167,1330]	[0.0093,0.0106]
	6 months	236160	[2282,2441]	[0.0097,0.0103]	[2249,2474]	[0.0095,0.0104]

Empirical Results

Backtesting (Christoffersen Test), the p-value.

	6 months	(236160)	3 months	(124800)	1 month	(40320)	1 week	(9600)	1 day	(1920)	1 hour	(240)
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour	0.0331	0.0083	0.0520	0.0375	0.0367	0.0165	0.0458	0.0291	0.0416	0.0083	0.0421	0.0020
(240)	*0.0250*	*0.0000*	*0.0183*	*0.001*	*0.0041*	*0.0001*	*0.00375*	*0.0001*	*0.0125*	*0.0025*	*0.0041*	*0.0000*
1 day	0.0388	0.0094	0.0161	0.0197	0.0354	0.0192	0.0401	0.0223	0.0234	0.0021		
(1920)	*0.0063*	*0.0021*	*0.0125*	*0.0036*	*0.0093*	*0.0021*	*0.0098*	*0.0005*	*0.03416*	*0.00416*		
1 week	0.0152	0.0013	0.0104	0.0113	0.0157	0.0160	0.0304	0.0025				
(9600)	*0.0354*	*0.0145*	*0.0813*	*0.0022*	*0.0146*	*0.0028*	*0.0137*	*0.0017*				
1 month	0.0544	0.0008	0.0667	0.0074	0.0415	0.0404						
(40320)	*0.0231*	*0.0011*	*0.0054*	*0.0012*	*0.0085*	*0.0019*						
3 months	0.0080	0.0017	0.0051	0.0007								
(124800)	*0.0031*	*0.0010*	*0.0054*	*0.0012*								
6 months	0.0207	0.0152										
(236160)	*0.0129*	*0.0083*										
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour	0.0041	0.0233	0.0125	0.0366	0.0458	0.0691	0.0000	0.0066	0.0083	0.0533	0.0000	0.0004
(240)	*0.0000*	*0.0250*	*0.0000*	*0.0167*	*0.0000*	*0.0042*	*0.0416*	*0.0520*	*0.0083*	*0.0125*	*0.0000*	*0.0143*
1 day	0.0014	0.0093	0.0014	0.0187	0.0348	0.0392	0.0182	0.0223	0.0036	0.0171		
(1920)	*0.0016*	*0.0063*	*0.0031*	*0.0313*	*0.0021*	*0.0093*	*0.0005*	*0.0094*	*0.0020*	*0.0119*		
1 week	0.0169	0.0384	0.0180	0.0215	0.0182	0.0271	0.0270	0.0304				
(9600)	*0.0012*	*0.0035*	*0.0021*	*0.0082*	*0.0020*	*0.0114*	*0.0015*	*0.0124*				
1 month	0.0154	0.0280	0.0070	0.0141	0.0181	0.0215						
(40320)	*0.0010*	*0.0031*	*0.0012*	*0.0054*	*0.0012*	*0.0084*						
3 months	0.0154	0.0180	0.0092	0.0099								
(124800)	*0.0083*	*0.0231*	*0.0009*	*0.0035*								
6 months	0.0036	0.0071										
(236160)	*0.0007*	*0.0029*										

Conclusion

- Based on a comparison of the goodness of fit criteria, the empirical evidence shows that the ARMA(1,1)-GARCH(1,1) (AG)model with Lévy fractional stable noise demonstrates better performance in modeling underlying asset dynamics.

- Considering the admissible VaR violations and violation frequencies for the prediction of 95% VaR, the AG model with Lévy fractional stable noise performs better than the other alternatives. For the prediction of 99% VaR, although all computed VaR values turns out to be conservative, the AG model with standard normal distribution has better performance among others, since they have violations close to the admissible VaR violations.

- The results of Christoffersen test suggest that the pattern of violation of our parametric VaR in the out-of-sample forecasting is consistently independent. We can observe that the VaR value calculated by the AG model with Lévy stable and fractional Lévy noise performs better than the alternative models because the p-values for rejecting the null hypothsis are less than that of the alternative models.

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