# Improved Forecasts of Higher-Order Comoments and Implications for Portfolio Selection

Volker Ziemann*

Edhec Risk and Asset Management Research Center

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**joint work with Lionel Martellini*

# Outline


1. Introduction
2. The Estimators
3. Empirical Results
4. Cutoff point analysis
5. Conclusion

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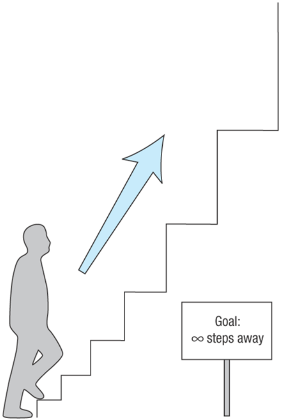
## Enhanced risk-return trade-off

- Kraus & Litzenberger (1976, JF)
- Harvey & Siddique (2000, JF)
- Dittmar (2002, JF)
- Mitton & Vorkink (2007, RFS)

# Portfolio construction and objective function

$$\mathrm{E}[\mathrm{U}(W)] \approx \mathrm{U}(\mathrm{E}(W)) + \frac{\mathrm{U}^{(2)}(\mathrm{E}(W))}{2}\mu^{(2)} + \frac{\mathrm{U}^{(3)}(\mathrm{E}(W))}{6}\mu^{(3)} + \frac{\mathrm{U}^{(4)}(\mathrm{E}(W))}{24}\mu^{(4)} + \dots$$



# This paper

## Problem

- How to suitable implement higher moments in portfolios choice problems
- Number of parameters increases exponentially with the number of assets

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For $N = 20$ assets one would need 45 years of monthly returns so that all moments and co-moments up to order 4 are well-defined by the sample observations.

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For $N = 20$ assets one would need 45 years of monthly returns so that all moments and co-moments up to order 4 are well-defined by the sample observations.

## Proposal

- Reducing parameter space
- Structural estimators
- Trade-off between estimation and specification risk

# Related literature

## Covariance matrix estimation

1. Structural Estimators
   - Sharpe (MS, 1963)
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## Higher moment parametrization

1. Parametric approach
   - Harvey et. al (WP, 2004)
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2. Time-conditional
   - Jondeau & Rockinger (JEDC, 2003)
   - Patton (JFEC, 2004)

# Our contributions

## Theoretical

- Derive suitable extensions of constant correlation and single factor estimators for higher moment tensors
- Extend the concept of optimal shrinkage intensities to structural higher moment estimates
- Analytic forms for higher order co-moments of multivariate variance-gamma processes

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## Empirical

- Extending the horse race methodology à la Chan et. al (1999) and Jagannathan & Ma (2003) to address the robustness issue and the stability of portfolios
- Calibrate the MVG process and evaluate cutoff-points for higher moment tensor estimations

# Main results

- Significant monetary utility gains when using structural estimators (2.86% to 4.32% p.a. in the base case)
- Portfolios built upon structural estimators are significantly more stable (between 28% and 32% turnover reduction)
- Shrinkage estimators slightly increase these gains
- Sample estimator portfolio allocations display more extreme weights and greater short interest
- These effects (stability and monetary utility gains) increase with smaller estimation windows, higher level of risk aversion and larger portfolios sizes
- Cutoff points increase with the moment dimension

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# Higher moment tensors

- We define higher moment tensors:

$$
\begin{aligned}
M_2 &= \mathbb{E}\left[(\mathrm{R}-\mathbb{E}(\mathrm{R}))(\mathrm{R}-\mathbb{E}(\mathrm{R}))\right] \\
M_3 &= \mathbb{E}\left[(\mathrm{R}-\mathbb{E}(\mathrm{R}))(\mathrm{R}-\mathbb{E}(\mathrm{R}))' \otimes (\mathrm{R}-\mathbb{E}(\mathrm{R}))'\right] \\
M_4 &= \mathbb{E}\left[(\mathrm{R}-\mathbb{E}(\mathrm{R}))(\mathrm{R}-\mathbb{E}(\mathrm{R}))' \otimes (\mathrm{R}-\mathbb{E}(\mathrm{R}))' \otimes (\mathrm{R}-\mathbb{E}(\mathrm{R}))'\right]
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with

$$
\begin{aligned}
\sigma_{ij} &= E\left[(R_i-\mu_i)(R_j-\mu_j)\right] \\
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## Example with 3 assets

$$M_2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

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\end{aligned}
$$

## Example with 3 assets

$$
M_3 = \left[\begin{array}{ccc|ccc|ccc}
s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\
s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\
s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333}
\end{array}\right] = \left[\begin{array}{c|c|c} S_{1jk} & S_{2jk} & S_{3jk} \end{array}\right]
$$

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## Example with 3 assets

$$
M_4 = \left[\begin{array}{ccc|ccc|ccc} K_{11jk} & K_{12jk} & K_{13jk} & K_{21jk} & K_{22jk} & K_{23jk} & K_{31jk} & K_{32jk} & K_{33jk} \end{array}\right]
$$

# Objective function

$$\min_{\omega} \left[ \frac{\gamma}{2}\mu^{(2)} - \frac{\gamma(\gamma+1)}{6}\mu^{(3)} + \frac{\gamma(\gamma+1)(\gamma+2)}{24}\mu^{(4)} \right]$$

$$\text{s.t. :} \qquad w' \, \mathbb{I}_n = 1$$

with

$$\boxed{\begin{aligned} \mu^{(2)} &= \omega' M_2 \omega \\ \mu^{(3)} &= \omega' M_3 (\omega \otimes \omega) \\ \mu^{(4)} &= \omega' M_4 (\omega \otimes \omega \otimes \omega) \end{aligned}}$$

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Table 0: Required number of parameters

| | M2 | M3 | M4 | Total |
|---|---|---|---|---|
| **N=5** | 15 | 35 | 70 | 120 |
| **N=20** | 210 | 1,540 | 8,855 | 10,605 |
| **N=100** | 5,050 | 171,700 | 4,421,275 | 4,598,025 |

# Constant correlation estimator

## Idea: Cauchy-Schwartz

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)E(Y^2)}$$

$$r_{ij}^{(1)} = \frac{E(\overline{R_i}\,\overline{R_j})}{\sqrt{\mu_i^{(2)}\mu_j^{(2)}}} \qquad r_{ij}^{(5)} = \frac{E(\overline{R_i}^2\,\overline{R_j}^2)}{\sqrt{\mu_i^{(4)}\mu_j^{(4)}}}$$

$$r_{ij}^{(2)} = \frac{E(\overline{R_i}\,\overline{R_j}^2)}{\sqrt{\mu_i^{(2)}\mu_j^{(4)}}} \qquad r_{ijk}^{(6)} = \frac{E(\overline{R_i}^2\,\overline{R_j}\,\overline{R_k})}{\sqrt{\mu_i^{(4)}E(\overline{R_j}^2\,\overline{R_k}^2)}}$$

$$r_{ij}^{(3)} = \frac{E(\overline{R_i}\,\overline{R_j}^3)}{\sqrt{\mu_i^{(2)}\mu_j^{(6)}}} \qquad r_{ijkl}^{(7)} = \frac{E(\overline{R_i}\,\overline{R_j}\,\overline{R_k}\,\overline{R_l})}{\sqrt{E(\overline{R_i}^2\,\overline{R_j}^2)E(\overline{R_k}^2\,\overline{R_l}^2)}}$$

$$r_{ijk}^{(4)} = \frac{E(\overline{R_i}\,\overline{R_j}\,\overline{R_k})}{\sqrt{\mu_i^{(2)}E(\overline{R_i}^2\,\overline{R_j}^2)}}$$

$$-1 \leq r^{(n)} \leq 1 \qquad \forall n = 1..7$$

# Constant correlation estimator

Consequently, the elements of $M_3$ and $M_4$ are given as:

$$
\begin{aligned}
\widehat{s}_{ij} &= \widehat{r}^{(1)}\sqrt{m_i^{(2)} m_j^{(2)}} \\
\widehat{s}_{iij} &= \widehat{r}^{(2)}\sqrt{m_i^{(4)} m_j^{(2)}} \\
\widehat{s}_{ijk} &= \widehat{r}^{(4)}\sqrt{m_k^{(2)} \widehat{r}^{(5)}\sqrt{m_i^{(4)} m_j^{(4)}}} \\
\widehat{k}_{iiij} &= \widehat{r}^{(3)}\sqrt{m_i^{(6)} m_j^{(2)}} \\
\widehat{k}_{iijj} &= \widehat{r}^{(5)}\sqrt{m_i^{(4)} m_j^{(4)}} \qquad \forall \quad i \neq j \neq k \neq l \\
\widehat{k}_{iijk} &= \widehat{r}^{(6)}\sqrt{m_i^{(4)} \widehat{r}^{(5)}\sqrt{m_j^{(4)} m_k^{(4)}}} \\
\widehat{k}_{ijkl} &= \widehat{r}^{(7)}\sqrt{\widehat{r}^{(5)}\sqrt{m_i^{(4)} m_j^{(4)}}\,\widehat{r}^{(5)}\sqrt{m_k^{(4)} m_l^{(4)}}}
\end{aligned}
$$

# Single factor estimator

## Assumptions

1. One driving factor explaining the asset return moments
2. Exposure with this factors determines cross-sectional differences
3. Residuals are independent (idiosyncratic risk)

$$
\boxed{\begin{aligned}
M_2 &= \mathbb{E}\left[(\beta\overline{\mathrm{F}}+\varepsilon)(\beta\overline{\mathrm{F}}+\varepsilon)'\right] \\
M_3 &= \mathbb{E}\left[(\beta\overline{\mathrm{F}}+\varepsilon)(\beta\overline{\mathrm{F}}+\varepsilon)'\otimes(\beta\overline{\mathrm{F}}+\varepsilon)'\right] \\
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\end{aligned}
$$

$$
\begin{aligned}
\widehat{M}_2 &= (\beta\beta')m_F^{(2)} && +\Sigma \\
\widehat{M}_3 &= (\beta\beta'\otimes\beta')m_F^{(3)} && +\Psi \\
\widehat{M}_4 &= (\beta\beta'\otimes\beta'\otimes\beta')m_F^{(4)} && +\Phi
\end{aligned}
$$

# Parameter space reduction

Table 1: Required number of parameters

| Panel A: N=5 | **M2** | **M3** | **M4** | **Total** |
|---|---|---|---|---|
| **Sample** | 15 | 35 | 70 | 120 |
| **Constant Correlation** | 6 | 18 | 19 | 27 |
| **Single Factor** | 11 | 11 | 17 | 23 |

| Panel B: N=20 | **M2** | **M3** | **M4** | **Total** |
|---|---|---|---|---|
| **Sample** | 210 | 1,540 | 8,855 | 10,605 |
| **Constant Correlation** | 21 | 63 | 64 | 87 |
| **Single Factor** | 41 | 41 | 62 | 83 |

| Panel C: N=100 | **M2** | **M3** | **M4** | **Total** |
|---|---|---|---|---|
| **Sample** | 5,050 | 171,700 | 4,421,275 | 4,598,025 |
| **Constant Correlation** | 101 | 303 | 304 | 407 |
| **Single Factor** | 201 | 201 | 302 | 403 |

# Optimal Shrinkage Intensities

$$\boxed{L(\alpha) = \| \alpha\Lambda + (1-\alpha)S - \Omega \|_F}$$

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## Asymptotic Estimator by Ledoit& Wolf

$$\alpha^* = \frac{1}{T}\frac{\pi - \rho}{\gamma}$$

- $\pi$: average asymptotic variances of sample estimator
- $\rho$: average asymptotic covariances between sample and structural estimator
- $\gamma$: average misspecification of structural estimators

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# Data & Methodology

- Assets listed on NYSE and AMEX (CRSP data base)
- Monthly returns from 05/1963 through 04/2006
- Only assets with valid data over the whole period are retained
- Small stocks are excluded (<20% percentile of market capitalization)
- Penny stocks are excluded (asset prices <5$)
- 100 random baskets with 20 assets are drawn from the total 255 available assets

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- At the end of April of each year higher moment tensors are estimated based on prior 60 months
- Optimal portfolio weights are obtained for a CRRA investor
- Portfolios are held until the next estimation (end of April of next year).
- 33 years of out-of-sample returns for 100 random portfolios are obtained for each estimator
- In the base case, we assume risk aversion $\lambda = 10$, 60 months in-sample horizon, 20 assets per portfolio and no transaction costs

# Empirical Results

Table 2: Mean squared estimation error

| Panel A: Co-variance | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 0.0333 | 0.0056 | 0.0233 | 0.0260 | 0.0329 | 0.0438 | 0.0496 |
| **Const. Correlation** | 0.0304 | 0.0055 | 0.0206 | 0.0233 | 0.0296 | 0.0416 | 0.0462 |
| **Single Factor** | 0.0310 | 0.0052 | 0.0227 | 0.0241 | 0.0305 | 0.0412 | 0.0461 |
| **Shrinkage to CC** | 0.0296 | 0.0056 | 0.0201 | 0.0227 | 0.0287 | 0.0406 | 0.0457 |
| **Shrinkage to SF** | 0.0306 | 0.0054 | 0.0219 | 0.0238 | 0.0300 | 0.0410 | 0.0464 |

| Panel B: Co-skewness | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 0.0171 | 0.0044 | 0.0096 | 0.0115 | 0.0165 | 0.0284 | 0.0309 |
| **Const. Correlation** | 0.0122 | 0.0044 | 0.0050 | 0.0073 | 0.0110 | 0.0237 | 0.0261 |
| **Single Factor** | 0.0108 | 0.0039 | 0.0058 | 0.0067 | 0.0099 | 0.0219 | 0.0229 |
| **Shrinkage to CC** | 0.0121 | 0.0044 | 0.0050 | 0.0073 | 0.0110 | 0.0237 | 0.0261 |
| **Shrinkage to SF** | 0.0108 | 0.0039 | 0.0058 | 0.0067 | 0.0099 | 0.0219 | 0.0229 |

| Panel C: Co-kurtosis | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 0.0202 | 0.0074 | 0.0098 | 0.0125 | 0.0184 | 0.0369 | 0.0432 |
| **Const. Correlation** | 0.0190 | 0.0083 | 0.0077 | 0.0101 | 0.0161 | 0.0402 | 0.0439 |
| **Single Factor** | 0.0146 | 0.0060 | 0.0065 | 0.0092 | 0.0128 | 0.0316 | 0.0337 |
| **Shrinkage to CC** | 0.0177 | 0.0074 | 0.0075 | 0.0096 | 0.0150 | 0.0371 | 0.0400 |
| **Shrinkage to SF** | 0.0147 | 0.0061 | 0.0064 | 0.0092 | 0.0128 | 0.0315 | 0.0341 |

# Empirical Results

Table 3: Optimal shrinkage intensities

Panel A: Shrinkage towards constant correlation

| | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **M2** | 0.7644 | 0.0690 | 0.5465 | 0.6313 | 0.7695 | 0.8657 | 0.9084 |
| **M3** | 0.9975 | 0.0055 | 0.9754 | 0.9825 | 1.0000 | 1.0000 | 1.0000 |
| **M4** | 0.7931 | 0.0682 | 0.6342 | 0.6775 | 0.7992 | 0.9081 | 0.9349 |

Panel B: Shrinkage towards single factor

| | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **M2** | 0.6145 | 0.0574 | 0.4681 | 0.5194 | 0.6174 | 0.7127 | 0.7413 |
| **M3** | 0.9755 | 0.0218 | 0.8777 | 0.9321 | 0.9820 | 0.9975 | 0.9993 |
| **M4** | 0.8805 | 0.0525 | 0.6941 | 0.7749 | 0.8943 | 0.9483 | 0.9703 |

# Empirical Results

Table 4: Monetary utility gains

| Panel A: 60 months in-sample | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Const. Correlation** | 2.86 | 3.09 | -3.24 | -1.99 | 2.68 | 7.63 | 15.91 |
| **Single Factor** | 4.32 | 2.90 | -1.08 | 0.33 | 3.75 | 9.50 | 17.09 |
| **Shrinkage to CC** | 3.56 | 2.81 | -2.27 | -0.55 | 3.05 | 7.97 | 16.67 |
| **Shrinkage to SF** | 4.41 | 2.57 | -0.28 | 1.11 | 4.00 | 9.05 | 17.35 |
| **Ex-post optimal** | 7.44 | 3.39 | 0.39 | 2.65 | 7.15 | 13.11 | 22.29 |
| Panel B: 36 months in-sample | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
| **Const. Correlation** | 9.77 | 6.45 | -1.54 | 0.16 | 8.98 | 20.06 | 45.83 |
| **Single Factor** | 11.72 | 6.57 | -0.36 | 2.40 | 10.82 | 22.55 | 48.53 |
| **Shrinkage to CC** | 10.39 | 6.37 | -1.53 | 0.48 | 9.63 | 20.34 | 45.27 |
| **Shrinkage to SF** | 11.34 | 6.36 | -0.93 | 2.16 | 10.34 | 21.57 | 46.47 |
| **Ex-post optimal** | 14.74 | 6.76 | 1.90 | 4.85 | 13.59 | 26.73 | 48.98 |

# Empirical Results

Table 5: Weight analysis

| Panel A: Short interest | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 0.5462 | 0.0585 | 0.4293 | 0.4465 | 0.5380 | 0.6410 | 0.7152 |
| **Const. Correlation** | 0.2160 | 0.0249 | 0.1539 | 0.1797 | 0.2137 | 0.2624 | 0.2870 |
| **Single Factor** | 0.2041 | 0.0259 | 0.1613 | 0.1699 | 0.2003 | 0.2625 | 0.2745 |
| **Shrinkage to CC** | 0.2029 | 0.0288 | 0.1481 | 0.1621 | 0.1984 | 0.2572 | 0.3045 |
| **Shrinkage to SF** | 0.2186 | 0.0304 | 0.1606 | 0.1775 | 0.2157 | 0.2786 | 0.2995 |
| **Ex-post optimal** | 0.1818 | 0.0542 | 0.0736 | 0.0981 | 0.1721 | 0.2685 | 0.3228 |
| Panel B: Positive weights | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
| **Sample** | 0.6182 | 0.0181 | 0.5621 | 0.5848 | 0.6205 | 0.6508 | 0.6606 |
| **Const. Correlation** | 0.6058 | 0.0334 | 0.5197 | 0.5477 | 0.6061 | 0.6652 | 0.6864 |
| **Single Factor** | 0.6642 | 0.0253 | 0.6015 | 0.6212 | 0.6621 | 0.7045 | 0.7212 |
| **Shrinkage to CC** | 0.6331 | 0.0285 | 0.5621 | 0.5818 | 0.6303 | 0.6765 | 0.7015 |
| **Shrinkage to SF** | 0.6847 | 0.0232 | 0.6167 | 0.6439 | 0.6871 | 0.7152 | 0.7348 |
| **Ex-post optimal** | 0.7045 | 0.0682 | 0.5000 | 0.6000 | 0.7000 | 0.8000 | 0.8500 |
| Panel C: Min weight | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
| **Sample** | -0.1727 | 0.0232 | -0.2463 | -0.2155 | -0.1700 | -0.1397 | -0.1276 |
| **Const. Correlation** | -0.0448 | 0.0052 | -0.0557 | -0.0540 | -0.0442 | -0.0360 | -0.0334 |
| **Single Factor** | -0.0659 | 0.0103 | -0.0949 | -0.0856 | -0.0653 | -0.0514 | -0.0485 |
| **Shrinkage to CC** | -0.0483 | 0.0063 | -0.0678 | -0.0613 | -0.0469 | -0.0402 | -0.0368 |
| **Shrinkage to SF** | -0.0757 | 0.0120 | -0.1151 | -0.0965 | -0.0750 | -0.0584 | -0.0546 |
| **Ex-post optimal** | -0.0689 | 0.0241 | -0.1439 | -0.1212 | -0.0661 | -0.0382 | -0.0241 |
| Panel D: Max weight | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
| **Sample** | 0.4033 | 0.0385 | 0.3397 | 0.3513 | 0.3971 | 0.4796 | 0.5040 |
| **Const. Correlation** | 0.3252 | 0.0439 | 0.2309 | 0.2648 | 0.3215 | 0.4042 | 0.5277 |
| **Single Factor** | 0.2845 | 0.0388 | 0.2131 | 0.2239 | 0.2833 | 0.3609 | 0.4689 |
| **Shrinkage to CC** | 0.3245 | 0.0428 | 0.2415 | 0.2639 | 0.3200 | 0.4021 | 0.5145 |
| **Shrinkage to SF** | 0.2958 | 0.0353 | 0.2298 | 0.2448 | 0.2939 | 0.3633 | 0.4632 |
| **Ex-post optimal** | 0.2797 | 0.0690 | 0.1357 | 0.1834 | 0.2743 | 0.3748 | 0.5571 |

# Empirical Results

Table 6: Turnover and optimal allocation

| Panel A: Realized turnover | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 2.00 | 0.12 | 1.75 | 1.80 | 2.00 | 2.20 | 2.38 |
| **Const. Correlation** | 1.47 | 0.09 | 1.24 | 1.33 | 1.47 | 1.60 | 1.70 |
| **Single Factor** | 1.37 | 0.07 | 1.20 | 1.26 | 1.37 | 1.49 | 1.55 |
| **Shrinkage to CC** | 1.41 | 0.09 | 1.22 | 1.27 | 1.41 | 1.56 | 1.72 |
| **Shrinkage to SF** | 1.35 | 0.08 | 1.18 | 1.22 | 1.35 | 1.51 | 1.57 |
| **Ex-post optimal** | 1.25 | 0.15 | 0.83 | 1.00 | 1.26 | 1.48 | 1.61 |

| Panel B: Required turnover | **Mean** | **Std** | **Min** | **5%** | **Med** | **95%** | **Max** |
|---|---|---|---|---|---|---|---|
| **Sample** | 1.75 | 0.16 | 1.42 | 1.52 | 1.74 | 2.03 | 2.12 |
| **Const. Correlation** | 1.14 | 0.10 | 0.95 | 0.98 | 1.13 | 1.33 | 1.41 |
| **Single Factor** | 1.09 | 0.10 | 0.91 | 0.95 | 1.09 | 1.28 | 1.30 |
| **Shrinkage to CC** | 1.10 | 0.09 | 0.94 | 0.96 | 1.10 | 1.25 | 1.38 |
| **Shrinkage to SF** | 1.11 | 0.09 | 0.91 | 0.99 | 1.10 | 1.24 | 1.33 |

# Outline

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# Multivariate gamma-process

- geometric brownian motion with stochastic time changes modeled by a gamma process $G$

$$A_t^{(n)} = A_0 \exp\left(\theta_n G_t + \sigma_n W_{G_t}^{(n)}\right), \qquad t \geq 0.$$

# Multivariate gamma-process

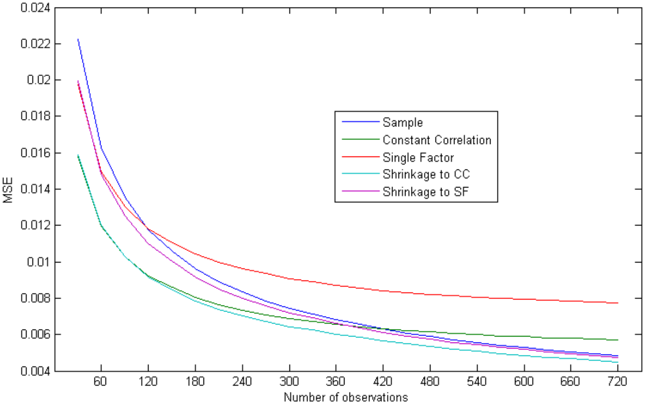
- geometric brownian motion with stochastic time changes modeled by a gamma process $G$

$$A_t^{(n)} = A_0 \exp\left(\theta_n G_t + \sigma_n W_{G_t}^{(n)}\right), \qquad t \geq 0.$$

- Allows for analytic formulas for higher moments
- Past research has concentrated on univariate gamma processes (see Madan & Seneta (1990, JB))
- Luciano & Schoutens (2006, QF) studied explicit forms for the covariance matrix and copulas
- We extend this literature to analytic forms for $M_2$, $M_3$ and $M_4$ when assets are MVG
- Trick: Use the characteristic function and its derivatives to derive the moment generating function

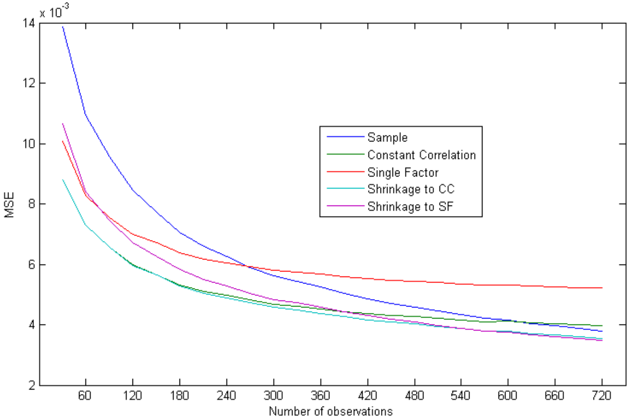
# Cutoff points - M2

Figure 1: Covariance estimation



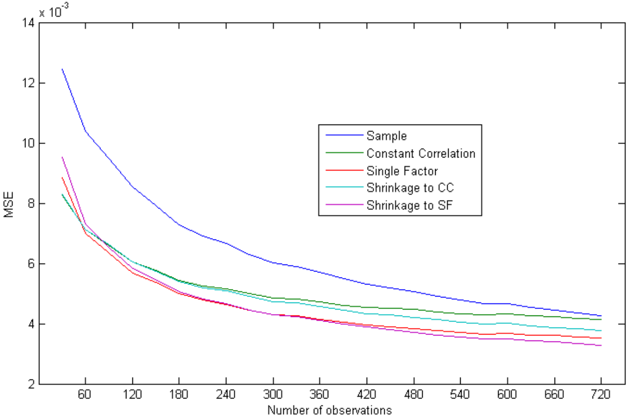
# Cutoff points - M3

Figure 2: Coskewness estimation



# Cutoff points - M4

Figure 3: Cokurtosis estimation



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# Conclusion

## Implications for portfolio choice

- Shrinkage effets prevail in portfolio choice with higher moments
- Important expected utility gains for CRRA type investors
- Significant turnover reduction with structural estimators
- Higher moment and co-moment calibration of Levy jump processes may have important implications in Option and Credit risk modeling

# Conclusion

## Implications for portfolio choice

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## Directions for future research

- Use of daily data may cut down the benefits (see Jagannathan & Ma (2003))
- Time-varying structural estimators versus rolling windows (DCC-Garch etc.)
- Bayesian portfolio choice