



Optimal Dynamic Hedging in Commodity Futures Markets with a Stochastic Convenience Yield

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Agenda

Motivation

- 1. The Economic Framework
- 2. Optimal Dynamic Strategies
- 3. Numerical Illustration
- 4. Future Research

Motivation

Copper



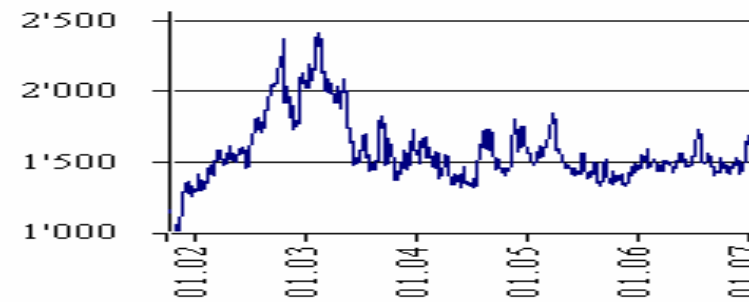
Gold



Oil-Brent



Cacao





Motivation (cont.)

- Commodity risk management
 - High fluctuations of commodity prices (metal, oil, agricultural commodities ...).
 - Use of futures contracts as hedging instruments.
- Does hedging the risk of storable commodities may deserve a specific treatment and require an appropriate model?



Motivation (cont.)

- Storable commodities differ from other financial assets by the convenience yield.
- Convenience yield: benefit accruing to the owner of a physical commodity as opposed to the owner of the futures contract.
- It is widely recognized that market prices of risk (MPR) evolve randomly over time (see Brennan, 1958; Litzenberger and Rabinowitz, 1995; Routledge et al. 2000).
- Moreover, stochastic MPR may be an additional source of mean-reversion in spot prices (Casassus and Collin-Dufresne, 2005).



Motivation (cont.)

- Main objective of our paper: what are the consequences of some characteristics of storable commodities (convenience yield, stochastic MPR) on the unconstrained investor's optimal demand.
 - The unconstrained investor can freely trade on the primitive assets (spot commodity and other risky assets).
- What is the role played by the futures contract and the primitive assets?



Motivation (cont.)

- Relevant literature
 - Constrained investor: Adler and Detemple, 1988; Duffie and Jackson, 1990; Briys et al. 1990; Duffie and Richardson, 1991; Lioui et al. 1996; Lioui and Poncet, 2001.
 - Unconstrained investor: Breeden, 1984.
 - Futures contracts have an instantaneous maturity and are perfectly correlated with the underlying commodity;
 - The convenience yield is not explicitly modeled.



Motivation (cont.)

- Main results:
 - In contrast to Breeden's paper, both the futures contract and the primitive assets may be used as hedging instruments.
 - Stochastic MPR induce stochastic speculative components, which can be separated in two terms related to the futures contract and the primitive assets respectively. Mean-reversion in MPR determines investor's speculative position.
 - New hedging terms appear due to the random behavior of the MPR. They can be decomposed into as many components as there are state variables.



Motivation (cont.)

- The interaction between the MPR associated with the state variables determine the investor's optimal demand, which evolves as a function of the level of these variables.
- The convenience yield turns out to be the crucial variable (in addition to the spot commodity) having the strongest impact on the investor's position.
- Our model has also important implications for practical use:
 - Speculative and hedging terms may be computed in a recursive way;
 - An investor is able to exactly assess the impact of each state variable on her optimal demand.



1. The Economic Framework

- State variables

- The spot commodity asset (log):

$$dX(t) = \left[r(t) - \delta(t) + \sigma_S \lambda_S(\delta, X) - \frac{1}{2} \sigma_S^2 \right] dt + \sigma_S dz_S(t); X(t) = \ln S(t)$$

- The short-term interest rate (Heath-Jarrow-Morton/Extended Vasicek process):

$$dr(t) = \alpha [\theta(t) - r(t)] dt + \sigma_r \left[\rho_{Sr} dz_S(t) + \sqrt{1 - \rho_{Sr}^2} dz_u(t) \right]$$

- The convenience yield:

$$d\delta(t) = \kappa (\bar{\delta} - \delta(t)) dt + \sigma_\delta \left[\rho_{S\delta} dz_S(t) + \rho_{u\delta} dz_u(t) + \rho_{v\delta} dz_v(t) \right]$$

$$\rho_{u\delta} = \frac{\rho_{r\delta} - \rho_{Sr} \rho_{S\delta}}{\sqrt{1 - \rho_{Sr}^2}}$$

$$\rho_{v\delta} = \frac{\sqrt{1 - \rho_{Sr}^2 - \rho_{S\delta}^2 - \rho_{r\delta}^2 + 2\rho_{Sr} \rho_{S\delta} \rho_{r\delta}}}{\sqrt{1 - \rho_{Sr}^2}}$$



1. The Economic Framework (cont.)

- Financial assets

- A riskless asset $\beta(t) = \exp\left\{\int_0^t r(s)ds\right\}$

- The spot commodity asset:

$$\frac{dS(t)}{S(t)} = [r(t) - \delta(t) + \sigma_S \lambda_S(\delta, X)]dt + \sigma_S dz_S(t)$$

- A discount bond:

$$\frac{dB(t, T_B)}{B(t, T_B)} = [r(t) - \sigma_B(t, T_B) \lambda_r(r)]dt - \sigma_B(t, T_B) \left[\rho_{sf} dz_S(t) + \sqrt{1 - \rho_{sf}^2} dz_u(t) \right]$$

- The futures price:

$$\frac{dH(t, T_H)}{H(t, T_H)} = \mu_H(t, T_H)dt + \sigma_{HS} dz_S(t) + \sigma_{Hu}(t, T_H) dz_u(t) + \sigma_{Hv}(t, T_H) dz_v(t)$$



1. The Economic Framework (cont.)

- MPR are affine in state variables so that the opportunity set is stochastic:

$$\lambda_X(S(t), \delta(t)) = \lambda_{X0} + \lambda_{XX}X(t) + \lambda_{X\delta}\delta(t)$$

$$\lambda_r(r(t)) = \lambda_{r0} + \lambda_{rr}r(t)$$

$$\lambda_\delta(\delta(t)) = \lambda_{\delta0} + \lambda_{\delta\delta}\delta(t)$$



1. The Economic Framework (cont.)

- The investor maximizes the expected constant relative risk aversion (CRRA) utility function of his (her) lifetime consumption and final wealth:

$$\max_{\{c(t), W(T_I)\}} \mathbb{E} \left[\int_t^{T_I} \frac{c(s)^{1-\gamma}}{1-\gamma} ds + \frac{W(T_I)^{1-\gamma}}{1-\gamma} \middle| F_t \right]$$

$$\text{s.t. } \frac{W(t)}{G(t)} = \mathbb{E} \left[\int_t^T \frac{c(s)}{G(s)} ds + \frac{W(T_I)}{G(T_I)} \middle| F_t \right]$$

$c(t)$ and $W(T_I)$ represent consumption at time t and the agent's terminal wealth respectively, $G(t)$ is the numéraire or growth optimal portfolio.

$$W(t) = \theta_S(t)S(t) + \theta_B(t)B(t, T_B) + \theta_\beta(t)\beta(t) + M(t)$$

$$M(t) = \int_0^t \exp \left\{ \int_u^t r(v) dv \right\} \theta_H(u, T_H) dH(u, T_H)$$



1. The Economic Framework (cont.)

- The solution to the investor's consumption-wealth problem may be simplified by operating an appropriate change of probability measure specific to the CRRA utility function.

$$\xi_\gamma(t, T_I) \equiv \frac{dP^{(\gamma, T_I)}}{dP} \Big|_{F_t} = \exp \left\{ -\frac{(\gamma-1)^2}{2\gamma^2} \int_0^t \|\lambda(u) - \sigma_B(t, T_I)\|^2 du - \frac{\gamma-1}{\gamma} \int_0^t [\lambda(u) - \sigma_B(u, T_I)]' dz(u) \right\}$$

- The numéraire is:

$$\frac{dN(t, T_I)}{N(t, T_I)} = \left[r(t) + \lambda(t) \left[\frac{1}{\gamma} \lambda(t) + \left(1 - \frac{1}{\gamma} \right) \sigma_B(t, T_I) \right] \right] dt + \left[\frac{1}{\gamma} \lambda(t) + \left(1 - \frac{1}{\gamma} \right) \sigma_B(t, T_I) \right] dz(t)$$



1. The Economic Framework (cont.)

- The optimal consumption and wealth at any date t :

$$c(t)^* = \frac{W(t)^*}{\Phi(\gamma, t, T_I)} \quad W(t)^* = \zeta^{-\frac{1}{\gamma}} G(t)^{\frac{1}{\gamma}} \Phi(\gamma, t, T_I)$$

$$\Phi(\gamma, t, T_I) = \left[\int_t^{T_I} \varphi(\gamma, t, s) ds + \varphi(\gamma, t, T_I) \right] \quad \varphi(\gamma, t, T_I) = B(t, T_I)^{\frac{1}{\gamma}} B_\gamma(t, T_I)$$

$$B_\gamma(t, T_I) = E_t^{(P^{\gamma, T_I})} \left[\exp \left\{ - \int_t^{T_I} \frac{\gamma - 1}{2\gamma^2} \|\lambda(u) - \sigma_B(u, T_I)\|^2 du \right\} \right]$$

ζ is the Lagrangian associated with the static program.



2. Optimal Dynamic Strategies

- The investor's optimal demand for risky assets consists in three terms.
 - A mean-variance speculative portfolio:

$$\pi^{MV}(t) = \begin{pmatrix} \pi_H^{MV}(t) \\ \pi_B^{MV}(t) \\ \pi_S^{MV}(t) \end{pmatrix} = \frac{1}{\gamma} \Sigma^{-1} \sigma \lambda(t) = \frac{1}{\gamma} \Sigma^{-1} \begin{bmatrix} \mu_X(t) - r(t) \\ \mu_B(t) - r(t) \\ \mu_H(t) \end{bmatrix}$$

- A hedging term related to the fluctuations of the interest rates;

$$\pi^{HIR}(t) = \left(1 - \frac{1}{\gamma}\right) \Sigma^{-1} \sigma \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_I)} \sigma_B(t, s) ds + \frac{\varphi(\gamma, t, T_I)}{\Phi(\gamma, t, T_I)} \sigma_B(t, T_I) \right]$$

- A hedging term stemming from the evolution of the MPR.

$$\pi^{HMPR}(t) = \begin{pmatrix} \pi_H^{HMPR}(t) \\ \pi_B^{HMPR}(t) \\ \pi_S^{HMPR}(t) \end{pmatrix} = \Sigma^{-1} \sigma \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_I)} \sigma_{B_\gamma}(t, s) ds + \frac{\varphi(\gamma, t, T_I)}{\Phi(\gamma, t, T_I)} \sigma_{B_\gamma}(t, T_I) \right]$$



2. Optimal Dynamic Strategies (cont.)

- To take a step forward, two replicable assets are introduced in the economy. They reflect idiosyncratic risks:
 - The first is associated with the specific risk of the interest rate;
 - The second asset is related to that of the convenience yield.

$$\frac{dB_u(t, T_B)}{B_u(t, T_B)} = \mu_{Bu}(t, T_B)dt - \rho_{ur}\sigma_B(t, T_B)dz_u(t)$$

$$\frac{dH_v(t, T_H)}{H_v(t, T_H)} = \mu_{Hv}(t, T_H)dt + \sigma_{Hv}(t, T_H)dz_v(t)$$



2. Optimal Dynamic Strategies (cont.)

- The optimal mean-variance proportions may be couched in a recursive way:

$$\pi_H^{MV}(t) = \frac{1}{\gamma} \frac{\mu_{Hv}(t, T_H) - r(t)}{\sigma_{Hv}^2(t, T_H)}$$

$$\pi_B^{MV}(t) = \frac{1}{\gamma} \left[\frac{\mu_{Bu}(t, T_B) - r(t)}{\sigma_{Bu}(t, T_B)' \sigma_{Bu}(t, T_B)} - \frac{\Sigma_{HB_u}(t, T_H, T_B)}{\sigma_{Bu}(t, T_B)' \sigma_{Bu}(t, T_B)} \pi_H^{MV}(t) \right]$$

$$\pi_S^{MV}(t) = \frac{1}{\gamma} \left[\frac{\mu_S(t) - r(t)}{\sigma_S^2} - \frac{\Sigma_{SB}(t, T_B)}{\sigma_S^2} \pi_B^{MV}(t) - \frac{\Sigma_{HS}(t, T_H)}{\sigma_S^2} \pi_H^{MV}(t) \right]$$



2. Optimal Dynamic Strategies (cont.)

- The investor's speculative demands will be adjusted as functions of the state variables.
- The futures contract is used to speculate against the idiosyncratic risk of the convenience yield.
- The discount bond and the spot commodity span the specific risks of the spot rate and the (log) spot price respectively. Due to correlations among these assets and the futures contracts, they are adjusted by terms involving the usual covariance/variance ratios.
- The speculative proportions may be computed in a recursive way and depend on excess returns, variances and covariances, facilitating hence their use for practical considerations.



2. Optimal Dynamic Strategies (cont.)

- The optimal hedging demand may be decomposed in two parts.
 - The hedging proportions related to the short-rate:

$$\pi_H^{HIR} = \pi_S^{HIR} = 0$$

$$\pi_B^{HIR}(t) = \left(1 - \frac{1}{\gamma} \right) \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s) \sigma_B(t, s)}{\Phi(\gamma, t, T_I) \sigma_B(t, T_B)} ds + \frac{\varphi(\gamma, t, T_I) \sigma_B(t, T_I)}{\Phi(\gamma, t, T_I) \sigma_B(t, T_B)} \right]$$



2. Optimal Dynamic Strategies (cont.)

- The hedging proportions generated by the stochastic MPR.
 - They can be expressed in a recursive way for each asset:

$$\pi_H^{HMPR}(t) = \frac{1}{\sigma_{Hv}(t)} \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma v}(t, s) ds + \frac{\varphi(\gamma, t, T_I)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma v}(t, T_I) \right]$$

$$\pi_B^{HMPR}(t) = \frac{1}{\sigma_{Bu}(t)} \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma u}(t, s) ds + \frac{\varphi(\gamma, t, T_I)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma u}(t, T_I) \right] - \frac{\Sigma_{HBu}(t)}{\sigma_{Bu}(t)^2} \pi_H^{HMPR}(t)$$

$$\pi_S^{HMPR}(t) = \frac{1}{\sigma_S} \left[\int_t^{T_I} \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma S}(t, s) ds + \frac{\varphi(\gamma, t, T_I)}{\Phi(\gamma, t, T_I)} \sigma_{B,\gamma S}(t, T_I) \right] - \frac{\Sigma_{SB}(t)}{\sigma_S^2} \pi_B^{HMPR}(t) - \frac{\Sigma_{HS}(t)}{\sigma_S^2} \pi_H^{HMPR}(t)$$



2. Optimal Dynamic Strategies (cont.)

- They may be decomposed into three Merton-Breeden-like components, one for each and every state variable:

$$\pi^{HMPR}(t) = \pi^{HMPR_X}(t) + \pi^{HMPR_r}(t) + \pi^{HMPR_\delta}(t)$$

$$\pi^{HMPR_i}(t) = \Sigma^{-1} \sigma \bar{\sigma}_i \left[\int_t^{T_i} \Psi_{\gamma_i}(t, s) \frac{\varphi(\gamma, t, s)}{\Phi(\gamma, t, T_i)} ds + \Psi_{\gamma_i}(t, T_i) \frac{\varphi(\gamma, t, T_i)}{\Phi(\gamma, t, T_i)} \right]$$

$$i \in \{X(t), r(t), \delta(t)\}$$

$$\Psi_{\gamma_i}(t, T_i) = \frac{B_{\gamma_i}(t, T_i)}{B_\gamma(t, T_i)}$$



2. Optimal Dynamic Strategies (cont.)

- The hedging terms stemming from the MPR may be expressed in two different ways. First, they can be associated with the three assets and, like the speculative demands, they can be calculated in a recursive way.
- Second, it is possible to disentangle the hedging element related to each state variable from those associated with the other state variables.
- An investor has the possibility to measure the impact of each state variable on her optimal demand. She can therefore decide ex-post to include or not in the investment opportunity set some variables when her objective is to implement hedging strategies.



2. Optimal Dynamic Strategies (cont.)

- Contrary to Breeden's (1984) result, both the futures contract and the primitive assets will be employed to hedge the risk of the state variables.
- The risk of the convenience yield is entirely hedged by the futures contract. $\Psi_{\gamma_i}(t, T_I)$ assesses the sensitivity of the hedging demands to changes in $B_{\gamma}(t, T_I)$ resulting from a change in the state variables.



3. Numerical Illustration

- Numerical simulation of the three terms:
 - different degrees of relative risk aversion (0.7, 1, 3, 6);
 - the investment horizon varies from 0 to 2 years;
 - the futures contract and the discount bond mature one month and five years respectively after the end of the investor's horizon;
 - other parameters values.

r	δ	X	α	κ	$\bar{\delta}$	σ_S	σ_r	σ_δ	$f(0,t)$
0.04	0.07	4.6	0.25	1.5	0.05	0.35	0.01	0.25	0.04
λ_{x0}	λ_{r0}	$\lambda_{\delta 0}$	λ_{xx}	$\lambda_{x\delta}$	λ_{rr}	$\lambda_{\delta\delta}$	ρ_{Sr}	$\rho_{S\delta}$	$\rho_{r\delta}$
12	0.15	0.5	-2.5	-1.5	-6	-2	-0.15	0.7	-0.1

3. Numerical Illustration (cont.)

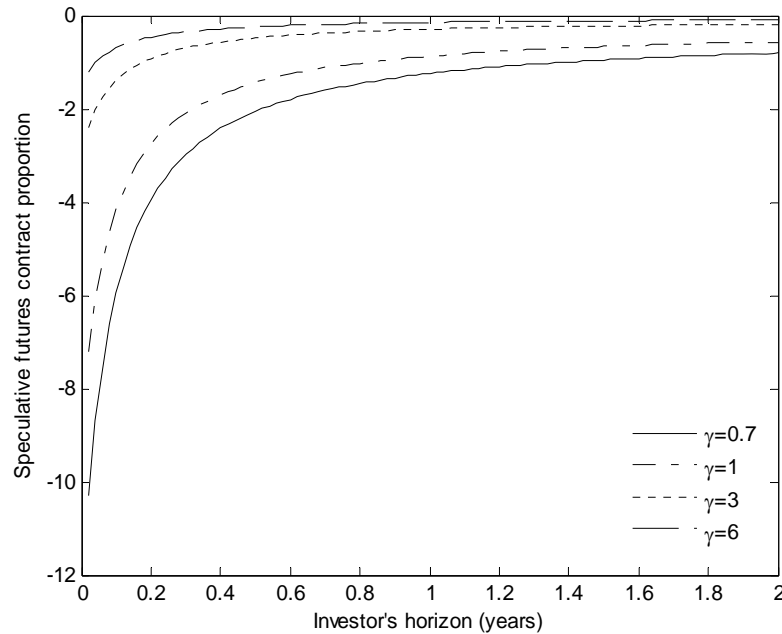


Fig. 1. *Speculative futures proportion varying with the investor's horizon.*

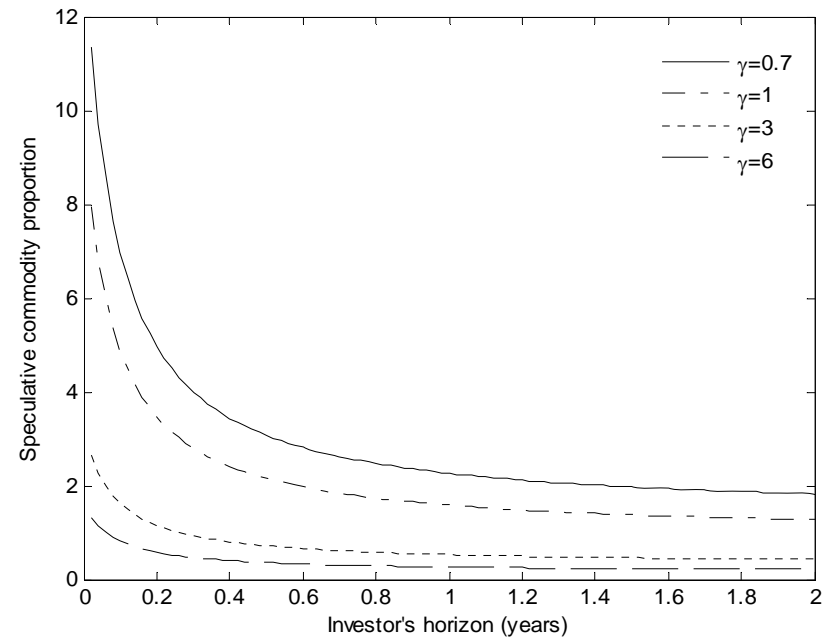


Fig. 2. *Speculative commodity proportion varying with the investor's horizon.*

3. Numerical Illustration (cont.)

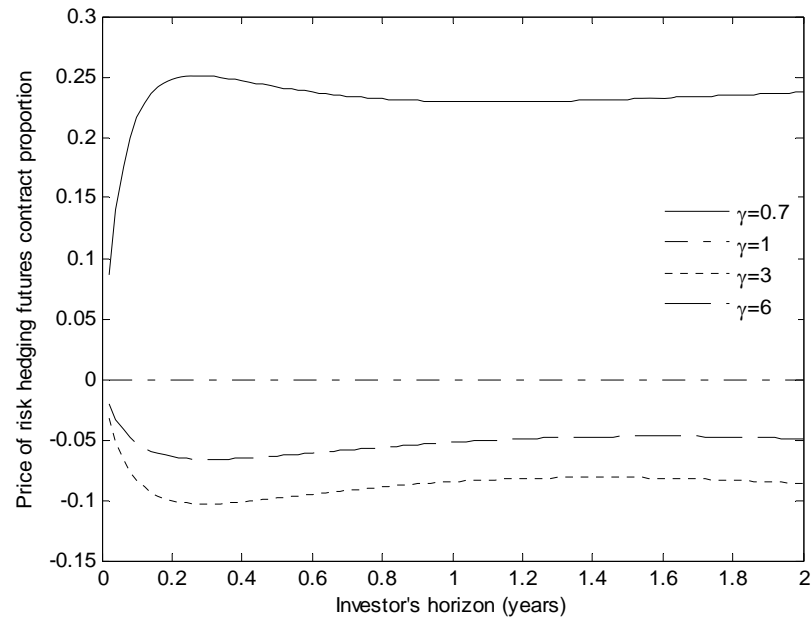


Fig. 3. *Price of risk hedging futures proportion varying with the investor's horizon.*

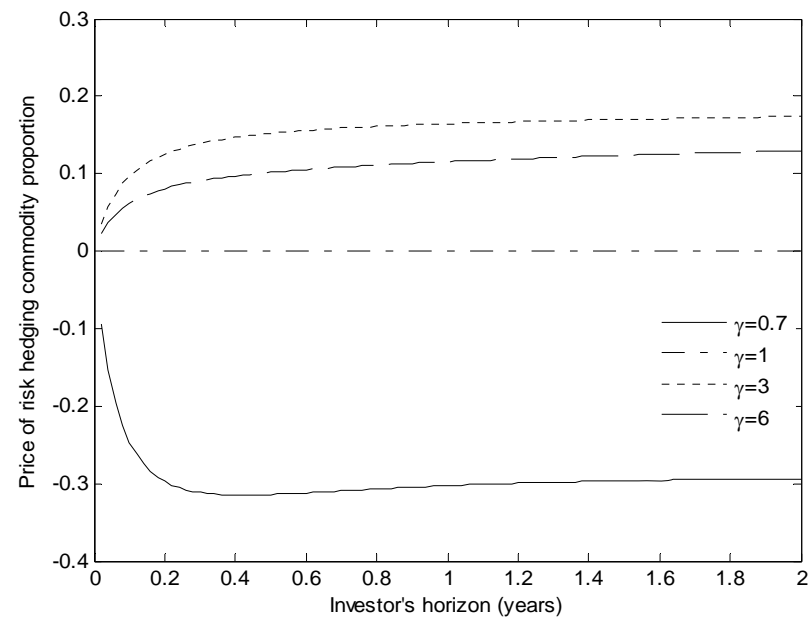


Fig. 4. *Price of risk hedging commodity proportion varying with the investor's horizon.*

3. Numerical Illustration (cont.)

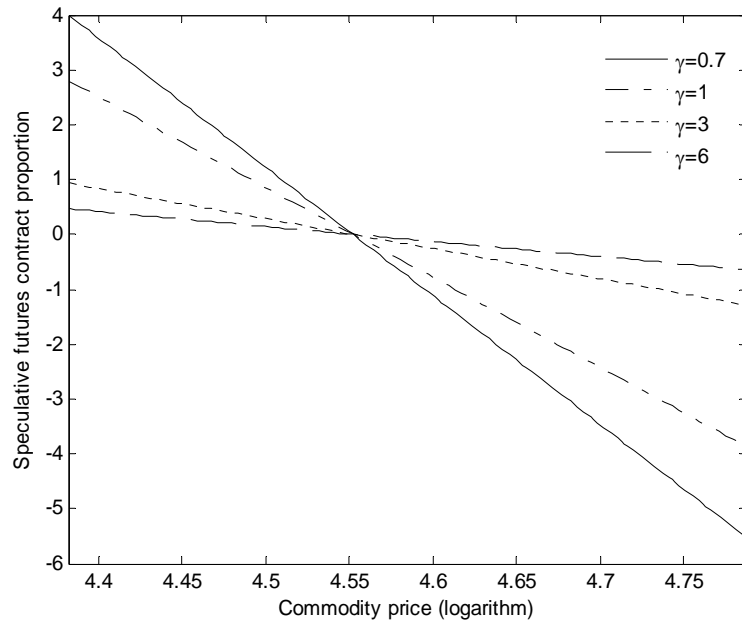


Fig. 5. *Speculative futures proportion varying with the commodity price.*

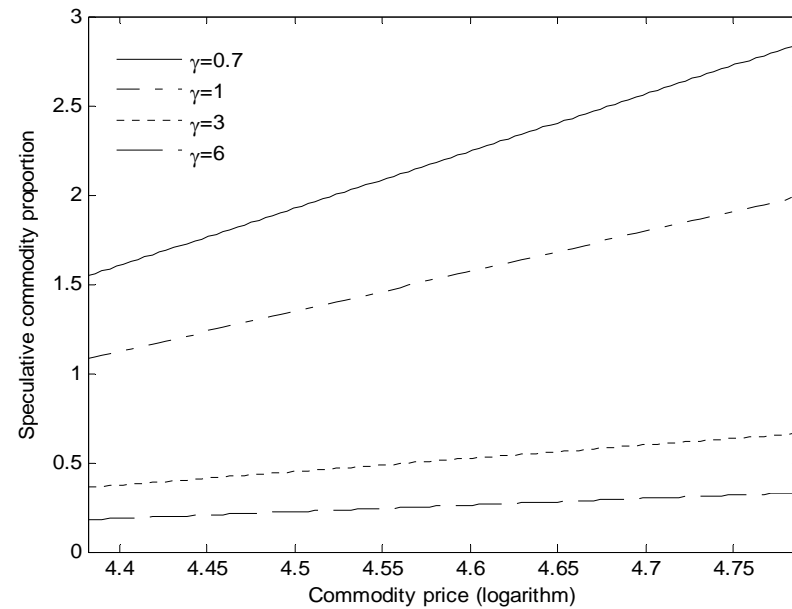


Fig. 6. *Speculative commodity proportion varying with the commodity price.*

3. Numerical Illustration (cont.)

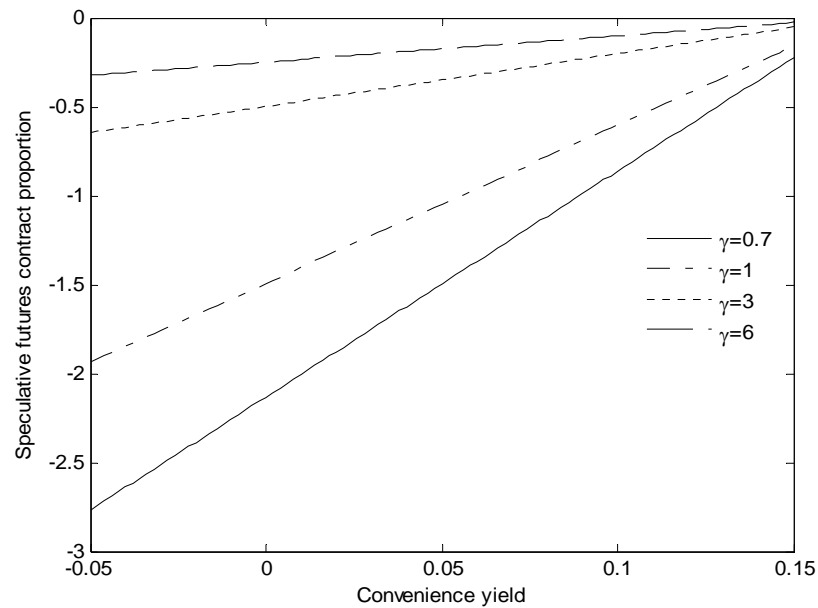


Fig. 7. *Speculative futures proportion varying with the commodity price.*

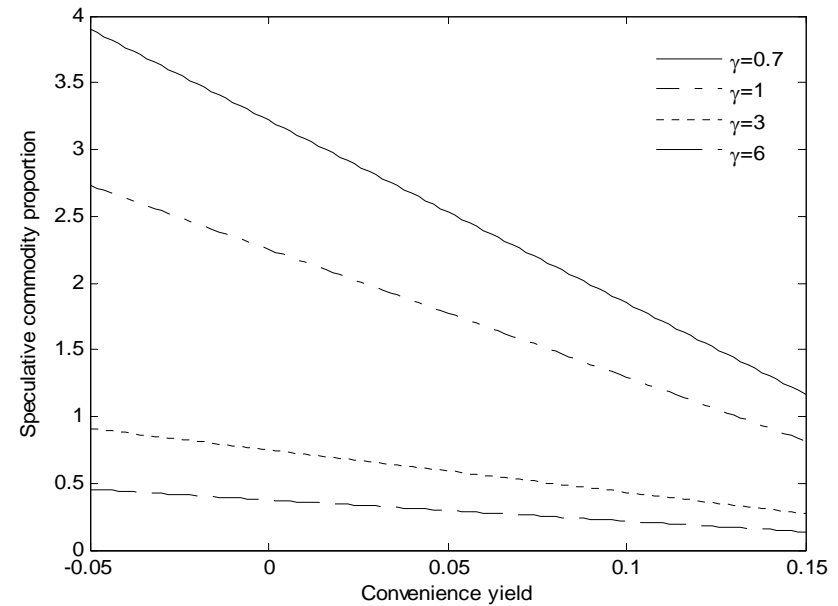


Fig. 8. *Speculative commodity proportion varying with the convenience yield.*

3. Numerical Illustration (cont.)

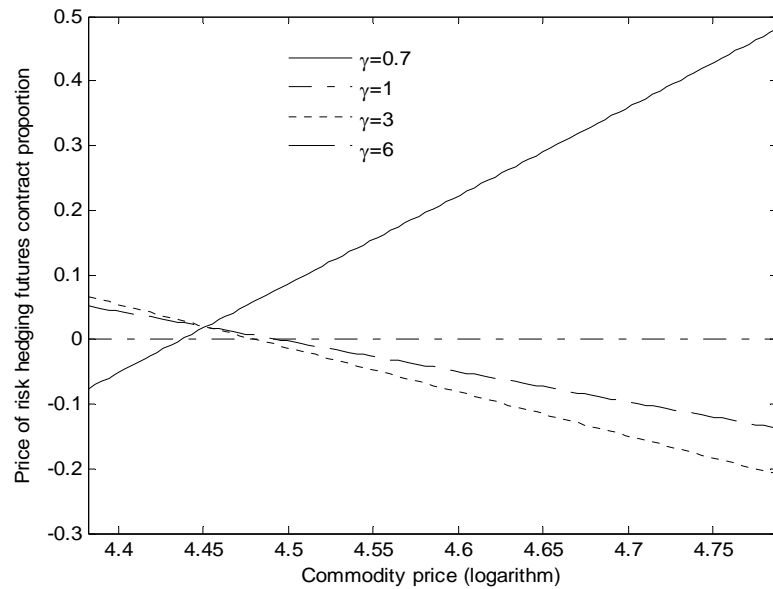


Fig. 9. Price of risk hedging futures proportion varying with the commodity price.

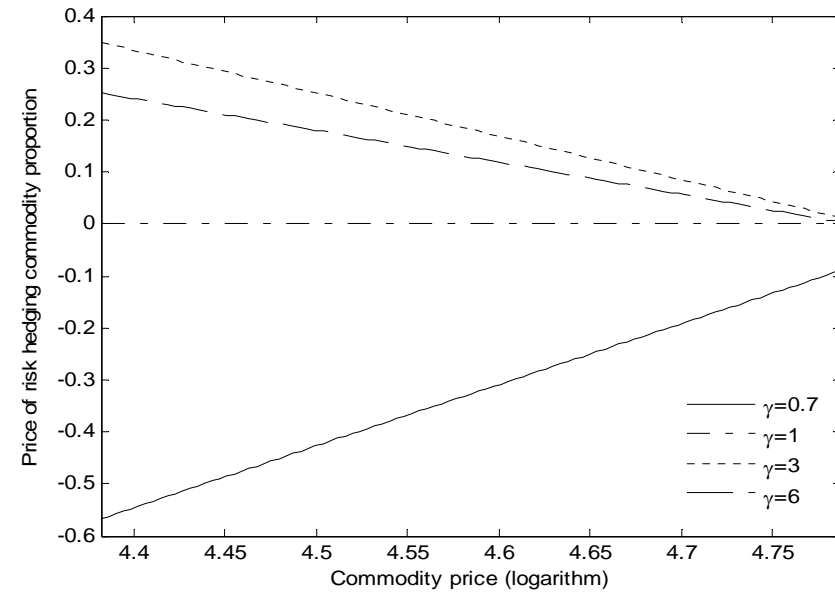


Fig. 10. Price of risk hedging commodity proportion varying with the commodity price.

3. Numerical Illustration (cont.)

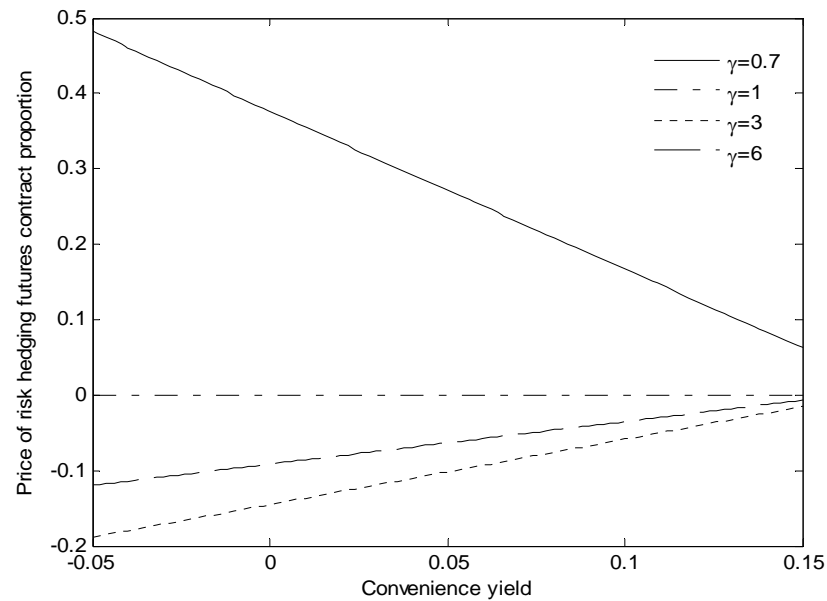


Fig. 11. Price of risk hedging futures proportion varying with the convenience yield.

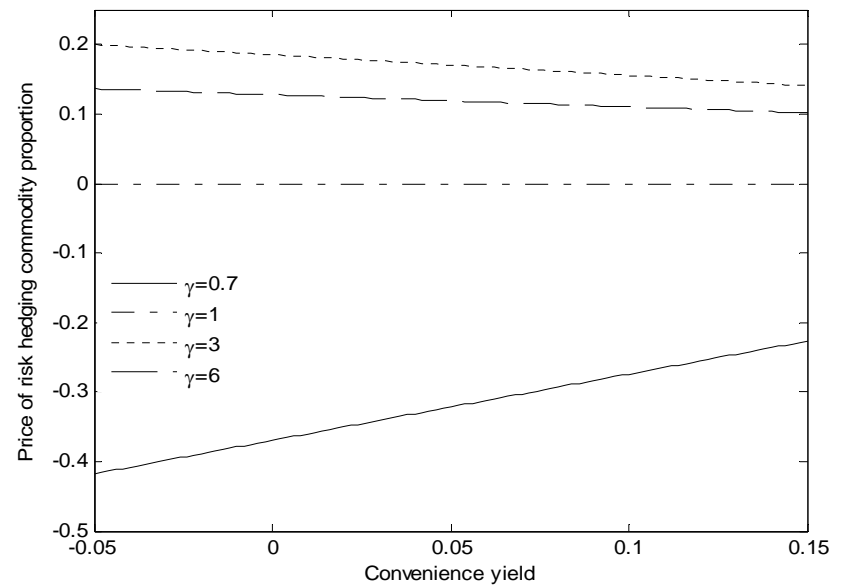


Fig. 12. Price of risk hedging commodity proportion varying with the convenience yield.



4.Future research

- Seasonality.
- Jumps in the spot commodity price and the convenience yield.
- Non-observability of the convenience yield.
- Constrained investor.