Should a Skeptical Portfolio Insurer use an Optimal or a Risk-Based Multiplier?

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Abstract
Following recent evidence of out-of-sample stock market return predictability, the authors aim to evaluate whether the potential benefits suggested by asset allocation theory can actually be captured in the real world using expected return estimates from a predictive system. The question is addressed in the context of an investor maximising the long-term growth rate of wealth under a maximum drawdown constraint, and comparing the optimal strategy using the predictive system with a similar risk-based allocation strategy, independent of expected return estimates. The authors find that the risk-based strategy implies nonetheless very variable and relatively high expected returns, and report important potential benefits in using the expected return estimates of the predictive system they used.

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EDHEC is one of the top five business schools in France. Its reputation is built on the high quality of its faculty and the privileged relationship with professionals that the school has cultivated since its establishment in 1906. EDHEC Business School has decided to draw on its extensive knowledge of the professional environment and has therefore focused its research on themes that satisfy the needs of professionals.

EDHEC pursues an active research policy in the field of finance. EDHEC-Risk Institute carries out numerous research programmes in the areas of asset allocation and risk management in both the traditional and alternative investment universes.
Most optimal asset allocation programs, including the pioneering mean-variance portfolio of Markowitz (1952)'s are a function of the assets' expected returns, which have been the holy grail of investment practitioners and financial economists for decades. The subject has been very controversial, and only until recently academic consensus started to move from the unpredictable random walk model towards a time-varying expected returns paradigm which implies some predictability must exist (see for instance Cochrane, 2008). However, most people recognise that estimating expected return remains challenging.

As a consequence, risk-based asset allocation programs have become increasingly popular amongst practitioners partly because they rely solely on risk-metrics and do not need expected returns estimates as an input. Examples of risk-based asset allocation models are the global minimum variance portfolio (see for instance Haugen and Baker, 1991), the equal risk contribution portfolio (ERC) of Maillard, Roncalli, and Teiletche (2010), the risk-based tangency portfolio of Martellini (2008), and portfolio insurance strategies using risk-based multipliers (examples of the latter are Bertrand and Prigent, 2002; Hamidi, Maillet, and Prigent, 2009a; Cont and Tankov, 2009).

Although risk-based asset allocation programs do not need expected return estimates as an input, from the standpoint of a rational investor they do imply some assumption about the expected return of assets. In that sense, not all risk-based asset allocations are strictly speaking “expected return agnostic”. For instance, Martellini (2008) assumes that expected returns are proportional to volatility, and Maillard et al. (2010) show that the ERC portfolio is mean-variance optimal assuming all assets’ Sharpe ratios and cross-asset correlations are equal. In a similar vein, hereafter we show how to extract the level of expected returns implied by the risk-based multiplier of a well-known portfolio insurance strategy. The implied expected returns found are quite variable and rather optimistic, relative to the historical long-term return average of the stock market.

The growing evidence in the academic literature of return predictability in the stock market poses the question of whether investors should ignore, or use some of the models available to implement their asset allocation programs. Hereafter we address that question in the case of an investor maximising the long-term growth rate of a portfolio under a maximum drawdown constraint. In particular, we compare the maximum drawdown control strategy, also known as time-invariant portfolio protection (TIPP) following Estep and Kritzman (1988), implemented using a risk-based multiplier on the one hand, and the growth-rate optimal multiplier of the strategy (following Grossman and Zhou, 1993; Cvitanic and Karatzas, 1995) on the other hand. The latter approach uses expected return estimates as an input.

Pástor and Stambaugh (2009) introduced a predictive system, which is a richer environment than the standard predictive regression to analyse the interactions amongst past realised returns, expected returns and predictors.¹ Unlike the predictive regression, the predictive system captures the mean-reverting dynamics of expected returns through an intuitive interaction between past realised returns and expected returns. This interaction results from two competing effects, the change effect and the level effect. The change effect captures for instance, the extend to which relatively low recent realised returns imply an increase in the expected return while the level effect the extend to which lower recent realised returns imply a decrease in the expected return. Pástor and Stambaugh’s (2009) predictive system may produce negative values for expected returns, which would imply a negative optimal multiplier. However, only positive multipliers makes sense for a portfolio insurer. Furthermore, if expected returns were negative over all horizons, prices would literally fall to zero, and in fact, any standard equilibrium model predicts positive expected returns. Recently, Bonelli and Mantilla-Garcia (2014) introduced a variation

¹ - The question of return predictability of the stock market has mostly been addressed within the linear predictive regression model. However, within the standard predictive regression framework, the ability of most widely known predictors to produce better forecasts than the prevailing historical mean out-of-sample have been questioned by Goyal and Welch (2008). On the other hand, out-of-sample predictive power improves when forecasts are combined (see Rapach, Strauss, and Zhou, 2010), the regression framework is modified (e.g. Dangl and Hulling, 2012) or economically motivated priors are imposed as in Campbell and Thompson (2006), and Pettenuzzo, Timmermann, and Valkanov (2012).
of the predictive system in which the modified expected return process is unlikely to produce negative values. This latter research paper highlights the benefits of the approach in terms of out-of-sample performance relative to the historical mean.

We find that using expected return estimates computed as in Bonelli and Mantilla-García (2014), to determine the optimal multiplier significantly improves the risk-adjusted performance of the drawdown control strategy compared to the equivalent risk-based allocation strategy. Indeed, the optimal strategy tends to be less risky, presents higher returns in most market configurations and its turnover is almost halved. Furthermore, unlike the risk-based strategy, its performance is more robust to the market boom and bust behaviour, thanks to the counter-cyclical dynamics of the estimated expected return process.

The strategy
The maximum drawdown (MDD) of an investment is defined as the largest value loss (in percentage) from a peak to a bottom ever observed at a given time. The MDD control strategy, consists of a dynamic allocation between cash and a performance-seeking portfolio that allows investors to limit drawdowns to a risk budget, $x \in (0, 1)$ set ex-ante. The risk budget is the maximum percentage loss the investor is willing to tolerate at any point in time. The strategy consists in allocating a proportion of wealth to the risky asset (or portfolio) at all times equal to,

$$\omega_S(t) = m_t \times \left(1 - \frac{F(t)}{A(t)}\right)$$

and the remaining wealth is invested in the riskless asset. Here $F$ denotes a Floor value process, $A$ the value of the portfolio and $m_t > 0$ is a key parameter that determines the dynamics of the strategy. This dynamic allocation rule is similar to the Constant Proportion Portfolio Insurance (Perold, 1986; Black and Jones, 1987; Perold and Sharpe, 1988), the main difference being the Floor formula and a varying (as opposed to constant) multiple. The MDD Floor is defined at every time $t$ as follows,

$$F(t) = (1 - x) \sup_{s \in [t, t]} A(s).$$

It is straightforward to show that, if the value of the portfolio is always above the MDD Floor, then the portfolio’s maximum drawdown is always lower than $(100 \times x)\%$. The Floor process is a strictly increasing function of time. Hence, for such a strategy to be able to insure the performance constraint $A(t) \geq F(t)$ for all $t$, the reserve asset must have a positive return at all times, thus bound to be cash.

Although the allocation of type (1) strategies theoretically changes continuously over time, trading happens in discrete time. Hence, we use a reallocation triggering rule that sets weights back to the current allocation target whenever the real portfolio exposure moves too far from its target as follows. Let the implied multiplier of type (1) strategies be defined as

$$\tilde{m}_t := \frac{\omega_S(t)}{c(t)}$$

where $c(t) := 1 - \frac{F(t)}{A(t)}$. No trading takes place whenever $\tilde{m}_t \in (m_t(1 - \tau), m_t(1 + \tau))$ and reallocations are triggered every time $\tilde{m}_t$ exits this no-trading band. In our illustrations below, we set $\tau = 0.2$ as in Hamidi, Maillet, and Prigent (2009b), which yields reasonable reallocation frequency and turnover figures.
The multipliers

The (positive) multiplier $m_t$ can be a constant or an adapted time-varying process. For a constant $m$ the allocation is to some extent a pro-cyclical (i.e., trend-following), as the strategy would sell shares of $S$ after experiencing losses and buy shares after a series of positive returns, unless its maximum exposure, $m_t x x$, is reached. However, Cvitanic and Karatzas (1995) (equation 6.4) showed that the optimal multiplier that maximises the long-term growth rate of the MDD strategy is,

$$m_t^* = \frac{\mu_S(t)}{\sigma_S^2(t)},$$

where $\mu_S(t)$ is the expected return of the risky asset $S$ in excess of the return of the riskfree asset, and $\sigma_S$ is the volatility of $S$. It is important to notice that, if expected excess returns are counter-cyclical, then the optimal multiplier can introduce a counter-cyclical (i.e., contrarian) component to the strategy.

While very efficient and widely known models are available to estimate and forecast the volatility of stocks (e.g. GARCH-type models, c.f. Engle, 2001), the estimation of expected returns has been much more controversial, and until recently the unpredictable random walk model was the standard in the profession. The former unpredictability consensus led many practitioners to use risk-based approaches to asset allocation. For instance, the multipliers of type (1) strategies are often set at an estimate of its discrete-time trading upper bound (e.g. Bertrand and Prigent, 2002; Cont and Tankov, 2009; Hamidi et al., 2009a), that is, the maximum value of the multiplier that would allow the strategy to respect its performance constraint and Floor for a given confidence level, even if trading happens in discrete time and asset prices present “jumps”. Bertrand and Prigent (2002), and Hamidi et al. (2009a) show that a discrete-time upper bound of the multiplier that maximises the exposure of the portfolio to the performance-seeking asset at all times is

$$\overline{m}_t = \frac{1}{ES^S_t(\alpha, h)},$$

where $ES^S_t(\alpha, h)$ denotes the expected- shortfall of the stock index $S$ with condence level $\alpha$ (often set to a high value, e.g. 99.99%) and a horizon of $h$ periods. The horizon $h$ corresponds to the time-frame it may pass before the manager can effectively reallocate assets, which for liquid assets might be one or two days at most. Thus, we set the target multiplier of the strategy $m_t$ is chosen such that $m_t(1 + \tau) = \overline{m}_t$.

The expected returns of equities over very short horizons are insignificant compared to the size of large percentiles of the return distribution. Thus, one can safely assume expected returns are equal to zero in the estimation procedure of the expected shortfall of equities over very short horizons (one or two days). Hence type (1) strategies using the maximum multiplier (4) are risk-based asset allocation programs.

There are different methods to estimate the expected shortfall in the academic literature. McNeil and Frey (2000) combine a GARCH model to filter the volatility of returns and use Extreme Value Theory (EVT) to model the tails of the distribution of the GARCH model residuals. This method has several advantages. First, it captures the market risk regimes (i.e., volatility clustering) as well as the natural fat-tails of stock returns. Second, the EVT approach is somewhat agnostic about the particular distribution of returns. Third, it is a parsimonious parametric model with relatively short computation time. McNeil and Frey (2000) find that their method produces more accurate tail risk estimates than alternative approaches using a Normal distribution to model the GARCH residuals or an unconditional EVT methodology.

Using McNeil and Frey (2000) methodology, we estimate the expected shortfall over a 2-days horizon at a confidence level of 99.99% based on daily returns of the S&P 500 index.
We use an initial calibration sample from January 1934 to December 1974, and perform daily estimations of ES from January 1975 to April 2014, using all data available at each point in time in the sample. Figure 1 presents the corresponding estimated maximum multiplier, used later in the out-of-sample historical Backtest section.

Figure 1: Optimal multiplier $m^*$ based on expected returns estimation and maximum multiplier $\bar{m}$ from January 1975 to April 2014.

**Predictive systems**

Hereafter we briefly summarise the predictive system used to estimate the expected returns required to calculate the optimal multiplier. We chose to use a predictive system that explicitly captures two economically motivated priors, i.e., the mean-reverting behaviour of expected returns and the idea that the latter are unlikely to be negative.

Recently Pástor and Stambaugh (2009) introduced a predictive system that incorporates a relationship between past realised returns and expected returns that explicitly captures two competing effects in the expected returns dynamics, called the *level effect* and the *change effect*. The level effect captures the extent to which observing relatively higher (lower) realised returns is a signal of higher (lower) expected returns, while the change effect captures the extent to which observing relatively higher (lower) returns is a signal of lower (higher) expected returns, given the mean reversion dynamics of the expected excess return.

Bonelli and Mantilla-García (2014) introduced a variation of Pástor and Stambaugh’s predictive system incorporating the economically motivated prior that expected returns are unlikely to be negative. Bonelli and Mantilla-García (2014) report out-of-sample predictability evidence using Pástor and Stambaugh’s original predictive system as well as an ulterior improvement using the modified predictive system. More precisely, Bonelli and Mantilla-García (2014) find that the mean square prediction error of the predictive system was significantly lower than the prediction error of the prevailing return average. They use the simpler predictive system without predictors in which expected returns are equal to a weighted average of past realised returns that incorporates the aforementioned effects in the expected returns dynamics as follows. The realised return $r$ at time $t+1$ is decomposed as

$$r_{t+1} = \mu_t + \epsilon_{t+1},$$

5 - McNeil and Frey’s (2000) ES estimation methodology requires a long series of historical data, as only the most extreme 5% of GARCH residuals available are used to estimate the parameters of the EVT tail distribution.
where $u_{t+1}$ is the "unexpected return". The *unobservable* expected return $\mu$ is a persistent mean-reverting process varying around the constant long-term return mean $E_r$, also driven by a random innovation process denoted $w$. If the expected return innovation $w$ is sufficiently negatively correlated with $u$, then the change effect prevails over the level effect.

Using the sample mean to estimate $E_r$ in the absence of predictors the expected return estimate of the system is given by

$$\hat{\mu}_t = \sum_{s=1}^{t} \kappa_s r_s$$

where

$$\kappa_s = \frac{1}{t} \left( 1 - \sum_{i=1}^{t} \omega_i \right) + \omega_s,$$

with $\sum_{s=1}^{t} \kappa_s = 1$, and $\omega_s = p_s (\beta - p_s)^s$, where $\beta$ is the autoregressive coefficient of order 1 of the expected return process $\mu$ and the parameter $p$ can be interpreted as the slope coefficient from the regression of $\mu_t$ on realised returns $r_t$, conditional on the sample information at time $t-1$. In Bonelli and Mantilla-García (2014) modified system, equations (6) and (7) have the same form than in the original predictive system of Pástor and Stambaugh (2009), but the terms $p_s$ have a different expression and depend on the level of $\mu_S$. The expression for $p$ is provided in the appendix. For further details on the predictive system we refer the reader to Bonelli and Mantilla-García (2014), and Pástor and Stambaugh (2009).

**Figure 2:** The figure presents estimates of $\kappa$, i.e. the weights on lagged realised returns in the expected return estimates when the unconditional mean return $E_r$ is estimated with the sample mean, $\beta = 0.9$, $\rho_{uw} = -0.85$. The weights in the figure corresponds to April 2014 for the S&P 500 index.

**Weights on lagged returns: $\kappa_s$**

![Weights on lagged returns](image)

We borrow from Bonelli and Mantilla-García (2014) the following point parameter values for the system parameters: the correlation between unexpected returns and innovations in expected returns is set to $\rho_{uw} = -0.85$, the persistence parameter $\beta = 0.9$ and a prior $R^2$ from the regression of $r_{t+1}$ on $\mu_t$ of 3%, which implies $\sigma^2 = 3\sigma^2 (\sigma_r$ is estimated as the sample standard deviations of realised returns). Figure 2 presents the resulting $\kappa$ of the system applied on quarterly returns of the S&P 500 index from January 1934 to April 2014. Figure 2 illustrates that, in this case, the change effect prevails, as most latest returns have a negative weight in the expected return estimate (6), while most older returns have a positive weight.

**Figure 3** (left-hand axis) presents the systems’ quarterly expected (excess) returns estimates of the S&P500 index from 1975 to 2014 (continuous line), recalculated every day in the sample using all past returns available at each estimation date. On the right-hand axis of the figure 6.
we present the daily log-cumulative return of the index (dotted line). A comparison between the two series illustrates the counter-cyclical dynamic in expected returns, which results from the prevailing change effect in the system. Indeed, the predictive system estimates tend to decrease during bull markets (anticipating returns reversals) and increase during bear markets (anticipating recovery), in contrast with the almost constant historical average (dashed line).

Using the predictive system estimates of $\mu_S$ for quarterly expected returns presented in Figure 3, and the quarterly variance series of $S$ calculated as the sum of the past 63 daily variance estimates from a EGARCH(1,1) model, we compute the optimal multiplier $m^*$ given by equation (3), and present it in Figure 1. From the figure, it is clear that the optimal multiplier is most of the time (80% of the days) lower than the maximum multiplier, thus the optimal multiplier does not constitute a riskier strategy than the one with the maximal multiplier. This result is in line with findings in Mantilla-Garcia (2014). Notice as well that the positivity assumption on expected returns yields an optimal multiplier that is always positive, implying only long positions in the performance seeking asset.

Figure 3: Quarterly expected excess return out-of-sample estimations for the S&P 500 using the predictive system, the prevailing historical average of returns, and the multiplier upper bound implicit estimate. The log cumulative returns are given on the right-hand axis. The sample period is January 1975 to April 2014.

The parameter set aforementioned is in line with empirical evidence in the academic literature. In particular Campbell (1991), and Binsbergen, Jules, and Koijen (2010) suggests $\rho_{uw}$ is close to -1 and the values for $\beta$ and $R^2$ are within the range of priors used by Pástor and Stambaugh (2009). The empirical evidence suggesting that expected returns in equities tend to increase following a series of negative realised returns and vice-versa, has a crucial implication for dynamic-allocation based portfolio insurance strategies. Indeed, in the presence of mean-reverting excess returns the optimal multiplier presents a counter-cyclical dynamic, as after a period of negative (resp. positive) realised returns, expected excess returns of the risky asset would increase (resp. decrease) and the optimal multiplier would be relatively higher (resp. lower) than at the beginning of the “bear” (resp. “bull”) market, anticipating the higher (resp. lower) returns. In stochastic simulations with mean reverting expected returns, Mantilla-Garcia (2014) reports significant potential benefits in using a growth-rate optimal multiplier, for a portfolio insurance strategy relative to the standard CPPI strategy with constant multiple in terms of performance and risk reduction.

Is the risk-based multiplier expected-return agnostic?

There is a fast growing literature presenting evidence of out-of-sample return predictability in the stock market. However, studies such as Goyal and Welch (2008) may casts doubts on the robustness of the predictability evidence and most people recognise that producing expected return estimates is a rather challenging task. The alternative to optimal asset allocation programs,
which use expected return estimates, are risk-based heuristics such as type (1) strategies using
the risk-based maximum multiplier (4) previously discussed. Although such a strategy does not use expected return estimates, the resulting allocation (and in fact any other heuristic asset allocation) can be reinterpreted from an optimal allocation perspective in order to deduce what would be the expected return estimate it implies. In other words, given an asset allocation, one may ask the question: if the objective would be to maximise the growth rate of the portfolio, what value of $\mu_S$ would yield such asset allocation?

Equating (3) and (4) yields the implied expected excess return of the risk-based multiplier,

$$\tilde{\mu}_S(t, h) = \frac{\sigma^2_S(t, h)}{ES^S_\alpha(t, h)}, \tag{8}$$

where $\sigma^2_S(t, h)$ is the variance of $S$ over a horizon of $h$ periods. For comparison purposes with the quarterly expected return estimates of the predictive system in the previous section, we aggregate $\tilde{\mu}_S(t, h)$ to obtain quarterly expected return estimates.

Figure 3 presents the implied quarterly expected returns of the risk-based maximum multiplier (4) previously presented in Figure 1. In order to obtain a variance corresponding to the same time horizon as the 2-days $ES$ estimates, the variance used in equation (8) is estimated using an EGARCH(1,1) model every day in the sample and then aggregated over the latest two days. Figure 3 presents the quarterly implied expected returns obtained by aggregating the 2-days $\tilde{\mu}_S$ over the latest 32 days, together with the quarterly prevailing historical average of returns.

It is clear from the Figure 3 that, in this case, the risk-based multiplier implies expected return estimates that are much higher than the ones implied by the historical mean and the predictive system most of the time. Furthermore, the implied expected return series presents much larger variations than the estimates from the predictive system. Hence, the risk-based approach represents a very bold depart from the non-predictability random walk hypothesis (i.e., the historical mean) and in this case it presents a rather optimistic view on expected returns.

**Historical Performance**

In order to evaluate whether the expected return estimates from the aforementioned predictive system would add value for an investor following a maximum drawdown control strategy, hereafter we compare the performance of the strategy using the optimal multiplier based on the expected return estimates and compare it with two benchmark portfolios, 1) a risk-based strategy using the maximum multiplier of equation (4) shown in Figure 1, and 2) a strategy using an optimal multiplier using the historical prevailing average return as the estimate for expected returns. Both strategies using the optimal multiplier formula are computed with the aforementioned quarterly (average daily) EGARCH(1,1) variance estimates.

We consider MDD strategies with a risk budget of $x = 25\%$, which corresponds to the drawdown level quoted by Grossman and Zhou (1993) in their study of the investment strategy for institutional (hedgefund) investors, and a "comparable level" to the calibrated drawdown limit of 31.5% by Lan, Wang, and Yang (2013). The riskless asset returns correspond to the 90-Day US T-Bill rate and the risky asset is the S&P500 total return index. We apply a variable transaction cost of 3 bps on each reallocation date and a lag of 1 day between the time the reallocation is triggered and its implementation, i.e. reallocations impact the returns between $t + 1$ and $t + 2$, where $t$ corresponds to the day when the reallocation is triggered and the estimation date of $m$. We impose an long-only (no leverage) constraint on the three strategies, i.e., $\omega_S(t) = \min (1, m_t x \sigma(t))$ for all $t$.  

8 - In the classic random walk model expected returns are constant over time and hence returns are unpredictable. Empirically, the prevailing historical mean of the stock market is virtually constant if enough data is used for its estimation.
We run the strategies over 5 samples ending in 04/2014 and starting in 01/1975, 01/1985, 01/1995, 01/2000, and 01/2005. The results of the out-of-sample backtests are reported in Table 1. As it can be observed from the table, the risk-adjusted performance of the optimal strategy using the predictive system improves considerably with respect to the risk-based maximal multiplier strategy. In all 5 sub-samples considered, the Sharpe and Calmar ratios of the strategy with \( m^* \) from the predictive system were higher than the one with \( \hat{m} \). The MDD of the optimal strategy over the longest sample was 20% compared to 24% for the risk-based strategy, and the average return of the optimal strategy was higher in all sub-samples except for the longest one (where the lag was 39 bps). Similarly, the optimal strategy using the predictive system presents higher annualised returns and Sharpe ratios than the strategy using the historical mean, in all sub-samples. Interestingly, the turnover and rebalancing frequency of the portfolio with the optimal multiplier is close to halved in all sub-samples relative to the risk-based strategy, i.e., the turnover of the optimal strategy using the predictive system is around 120% compared to circa 300% turnover for the risk-based strategy. Similarly, the average number of days between reallocation dates is circa 40 business days compared to about 20 days for the optimal and risk-based strategies respectively.

In general we observe that the counter-cyclical \( m^* \) of the predictive system allows the strategy to have a smoother performance over time. On the other hand, the \( \hat{m} \) portfolio performs better during the initial bull market periods, but it suffers from larger losses as observed after the bubble bursts in 2000 causing it to miss out most of the subsequent recovery period.

It is important to notice that all sub-periods start with a relatively long bull market, except for the one starting in January 2000. This period illustrates best the advantage of using the optimal multiplier over the risk-based approach or the optimal multiplier using the historical return mean as expected return estimate. Indeed, during the 2000-2014 period, we observe a large advantage of the optimal strategy over the risk-based strategy in terms of average return, risk-adjusted performance ratios and risk figures. In effect, the average return of the optimal strategy is more than twice, its Sharpe ratio almost 5 times, and its Calmar ratio 3 times the respective figures of the risk-based strategy.

Figure 4 presents the cumulative returns of the two portfolios along with their floors and allocations over the 2000-2014 period. At the start of the period, the predictive system estimated a relatively low expected return, due to the previous long bull market of the IT bubble. As a consequence, the initial allocation of the strategy to the S&P 500 index would have been 40% (with a \( m = 1.6 \)), compared to 100% for the risk-based strategy (with a \( m = 4.6 \)) and 67% for the historical average-based multiplier of 2.7.
Table 1: Statistics of MDD strategies using optimal multiplier based on the predictive system and the historical average of expected-returns, and a MDD strategy using the maximal multiplier. Strategies are computed over different sub periods between January 1975 and April 2014. The risky asset is the S&P 500 Index and the cash rate is the 90-Day US T-Bill.

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<th>Period: 01/1975-04/2014</th>
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<th>m* Hist.</th>
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<th>S&amp;P 500</th>
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<td>0.42</td>
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<th>S&amp;P 500</th>
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Figure 4: Panel A and B present the log-cumulative return and floors of the MDD strategies using the predictive system based multiplier and the strategy using the maximum multiplier, respectively. Panel C and D present the cash and equity allocations of corresponding strategies above them. The backtest period is January 2000 to April 2014.
Conclusion
The growing evidence in the academic literature of the (out-of-sample) predictability of the stock market poses the question of whether investors should use the expected return estimates available to determine their asset allocation. We use a variation of the predictive system of Pástor and Stambaugh (2009), introduced in Bonelli and Mantilla-García (2014), that incorporates the economically motivated priors that expected returns are positive, persistent and mean-reverting.

We use the expected returns estimates from the predictive system to compute the optimal multiplier of a portfolio insurance strategy controlling the maximum drawdown.

Empirically, we find that the optimal multiplier is lower than a risk-based maximal multiplier. We show that the risk-based multiplier implies expected return estimates with very large variations and most of the time with much larger levels than the prevailing historical mean of stock returns. In out-of-sample backtests we find that using the optimal multiplier improves the risk-adjusted performance of the strategy while reducing turnover and reallocation frequency. The optimal strategy using the predictive system also presents a more robust performance relative to the inception date of the strategy and across market boom and bust cycles.

Appendix
This appendix borrows some equations from Bonelli and Mantilla-García (2014) to provide enough details to implement the expected return estimates used in this paper. Although in this paper no predictors where used, the equations hereafter describes how to estimate the unobservable expected return process $\mu$ based on observations of realised returns $r$ and a set of predictors $x$.

The algorithm corresponds to an extended version of the Kalman filter (see Anderson and Moore (2012) Chap. 8) for the following state space model:

$$
\begin{align*}
r_{t+1} &= \mu_t + u_{t+1}, \\
\mu_{t+1} &= (1 - \beta)E_r + \beta \mu_t + g(\mu_t)w_{t+1}, \\
x_{t+1} &= (I - A)E_x + Ax_t + v_{t+1},
\end{align*}
$$

where $E_r$ is the unconditional expectation of $r$ and $\mu$, $\beta$ is an auto-regressive parameter, $g$ is a general function, $E_x$ is the unconditional expectation of $x$ and $A$ is a matrix with suitable dimensions containing the autoregressive coefficients and with eigenvalues lying inside the unit circle. Furthermore, the three innovation processes $u$, $v$, $w$ above are assumed to be correlated white-noise, independent and identically distributed across $t$ as,

$$
\begin{bmatrix}
u_t \\
w_t \\
w_{t+1}
\end{bmatrix} \sim \mathcal{N}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\sigma_u^2 & \sigma_{uw} & \sigma_{uw} \\
\sigma_{vu} & \Sigma_{uv} & \sigma_{uw} \\
\sigma_{wu} & \sigma_{uw} & \sigma_w^2
\end{bmatrix}.
$$

9. This includes $g(.) = \sqrt{1}$ corresponding to the particular case of the predictive system we use in the text, and $g(.) = 1$ corresponding the initial system of Pastor and Stambaugh (2009).
The following notations are used:

\[ z_t = \begin{bmatrix} r_t \\ x_t \end{bmatrix}, \quad a_t = \mathbb{E}(\mu_t|D_{t-1}), \quad b_t = \mathbb{E}(\mu_t|D_t), \quad f_t = \mathbb{E}(z_t|D_{t-1}), \quad P_t = \text{Var}(\mu_t|D_{t-1}), \]

\[ Q_t = \text{Var}(\mu_t|D_t), \quad R_t = \text{Var}(z_t|\mu_t, D_{t-1}), \quad S_t = \text{Var}(z_t|D_{t-1}), \quad G_t = \text{Cov}(z_t, \mu_t|D_{t-1}), \]

where \( D_t \) corresponds to the information available at time \( t \), i.e. \( D_t = (r_1, x_1, r_2, x_2, \ldots, r_t, x_t) \).

**Initialisation** We assume conditioning on the (unknown) parameters even if not explicitly specified and that \( D_0 \) denotes the null information set.

Assuming that \( \mu_1 \sim \mathcal{N}(E_r, V_{\mu}) \) and \( r_1 \sim \mathcal{N}(E_r, V_r) \), given \( V_x, V_{rx}, V_{\mu}, V_{x\mu} \) we have first

\[
\begin{align*}
a_1 &= E_r, \quad P_1 = V_\mu, \quad f_1 = [E_r \ E_x]', \quad S_1 = \begin{bmatrix} V_r & V_{rx} \\ V_{rx} & V_x \end{bmatrix}, \quad G_1 = [V_{rx} \ V_{x\mu}]', \\
R_1 &= S_1 - G_1 P_1^{-1}G_1', \\
Q_1 &= P_1 (P_1 + G_1' R_1^{-1} G_1)^{-1} P_1, \\
b_1 &= a_1 + P_1 (P_1 + G_1' R_1^{-1} G_1)^{-1} G_1' R_1^{-1} (z_1 - f_1). 
\end{align*}
\]

**Iteration** We use the extended Kalman filter algorithm to derive, for \( t = 2, \ldots, T \),

\[
\begin{align*}
a_t &= (1 - \beta)E_r + \beta \mathbb{E}(\mu_{t-1}|D_{t-1}) + \mathbb{E}(g(b_{t-1})w_t|D_{t-1}) = (1 - \beta)E_r + \beta b_{t-1}, \\
P_t &= \text{Var}((1 - \beta)E_r + \beta \mu_{t-1} + g(b_{t-1})w_t|D_{t-1}) \\
&= \beta^2 \text{Var}(\mu_{t-1}|D_{t-1}) + \text{Var}(g(b_{t-1})w_t|D_{t-1}) + 2\beta \text{Cov}(\mu_{t-1}, g(b_{t-1})w_t|D_{t-1}) \\
&= \beta^2 Q_{t-1} + g(b_{t-1})^2 \sigma_w^2. 
\end{align*}
\]

We have:

\[
S_t = \begin{bmatrix} \text{Var}(r_t|D_{t-1}) & \text{Cov}(r_t, x_t|D_{t-1}) \\ \text{Cov}(r_t, x_t|D_{t-1}) & \text{Var}(x_t|D_{t-1}) \end{bmatrix} = \begin{bmatrix} Q_{t-1} + \sigma_w^2 \\ \sigma_w \Sigma_{uv} \end{bmatrix},
\]

\[
G_t = \begin{bmatrix} G^1_t \\ G^2_t \end{bmatrix},
\]

with

\[
G^1_t = \text{Cov}(\mu_{t-1} + u_t, (1 - \beta)E_r + \beta \mu_{t-1} + g(b_{t-1})w_t|D_{t-1}) \\
= \beta Q_{t-1} + \text{Cov}(u_t, g(b_{t-1})w_t|D_{t-1}) \\
+ \beta \text{Cov}(u_t, \mu_{t-1}|D_{t-1}) + \text{Cov}(u_t, g(b_{t-1})w_t|D_{t-1}) \\
= \beta Q_{t-1} + g(b_{t-1}) \sigma_{uw},
\]

and

\[
G^2_t = \text{Cov}((I - A)E_x + Ax_{t-1} + v_t, (1 - \beta)E_r + \beta \mu_{t-1} + g(b_{t-1})w_t|D_{t-1}) \\
= \text{Cov}(v_t, g(b_{t-1})w_t|D_{t-1}) \\
= g(b_{t-1}) \sigma_{vw}.
\]

Finally

\[
G_t = \begin{bmatrix} \beta Q_{t-1} + g(b_{t-1}) \sigma_{uw} \\ g(b_{t-1}) \sigma_{vw} \end{bmatrix}. 
\]
The last terms are functions of those previously computed:

\[ R_t = S_t - G_t P^{-1} G_t', \quad (13) \]

\[ Q_t = P_t (P_t + G_t' R_t^{-1} G_t)^{-1}, \quad (14) \]

\[ f_t = \left[ \frac{E(\mu_{t-1}|D_{t-1})}{(I-A)E_x + Ax_{t-1}} \right] = \left[ \frac{b_{t-1}}{(I-A)E_a + Ax_{t-1}} \right]. \quad (15) \]

The filtering term \( b_t \) (i.e., the estimate of \( \mu \)) is given by

\[ b_t = a_t + P_t (P_t + G_t' R_t^{-1} G_t)^{-1} G_t' R_t^{-1} (z_t - f_t) = a_t + G_t' S_t^{-1} (z_t - f_t). \quad (16) \]

Let us denote

\[ [p_t, n_t'] = P_t (P_t + G_t' R_t^{-1} G_t)^{-1} G_t' S_t^{-1} = Cov(z_t', \mu_t|D_{t-1})[\text{Var}(z_t|D_{t-1})]^{-1} \]

\[ = [\beta Q_{t-1} + g(b_{t-1})\sigma_{uvw}, g(b_{t-1})\sigma_{uuv}] \left[ \frac{Q_{t-1} + \sigma^2}{\sigma_{uu} \Sigma_{ww}} \right]^{-1}. \quad (17) \]

If \( E_r \) is replaced by the sample mean, it can be shown that the estimate of \( b_t \) is,

\[ \hat{b}_t = \sum_{s=1}^t \kappa_s r_s + \sum_{s=1}^t \delta_s' v_s, \quad (18) \]

with

\[ \kappa_s = \frac{1}{t} \left( 1 - \sum_{t=1}^t \omega_t \right) + \omega_s, \quad (19) \]

and \( \sum_{s=1}^t \kappa_s = 1 \), where,

\[ \omega_s = \begin{cases} (\beta - p_t)(\beta - p_{t-1}) \ldots (\beta - p_{t+1})p_s, & \text{for } s < t \\ p_s, & \text{for } s = t. \end{cases} \]

and

\[ \delta_s = \begin{cases} (\beta - p_t)(\beta - p_{t-1}) \ldots (\beta - p_{t+1})n_s, & \text{for } s < t \\ n_s, & \text{for } s = t. \end{cases} \]

References


• Pettenuzzo, D., A. Timmermann, and R. Valkanov (2012). Forecasting stock returns under economic constraints. Available at SSRN.

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