Towards Conditional Risk Parity — Improving Risk Budgeting Techniques in Changing Economic Environments

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The present publication was produced as part of the “Asset Allocation Solutions” research chair at EDHEC-Risk Institute, in partnership with Lyxor Asset Management. This chair is examining performance portfolios with improved hedging benefits, hedging portfolios with improved performance benefits, and inflation risk and asset allocation solutions.

The current paper, “Towards Conditional Risk Parity – Improving Risk Budgeting Techniques in Changing Economic Environments” looks at the topical issue of risk parity. It has become increasingly apparent that a portfolio that seems to be well-balanced in terms of dollar contributions can be extremely concentrated in terms of risk contributions because of differences in volatility and pairwise correlation levels amongst the constituents.

In this context, risk parity, which assigns the same contribution to portfolio risk to all assets, has become an increasingly popular risk management methodology within and across asset classes. While intuitively appealing, this approach suffers from one major shortcoming, namely the fact that it is not explicitly sensitive to changes in market conditions. In particular, using the risk parity approach in an asset allocation context inevitably leads to a substantial overweighting of bonds versus equities, which might be a concern in a low bond yield and high dividend yield economic environment.

In this paper, we introduce three distinct conditional risk parity strategies, explicitly designed to optimally respond to changes in state variables that have been used in the literature as proxies for the stochastically time-varying opportunity set. In an empirical analysis, we document the superiority in various economic regimes of such conditional risk parity strategies with respect to standard unconditional risk parity techniques.

I would like to thank Professor Lionel Martellini, Dr Vincent Milhau and Dr Andrea Tarelli for their work on this research, and Laurent Ringelstein and Dami Coker for their efforts in producing the final publication.

I would also like to extend our warmest thanks to Lyxor Asset Management for their support of this research chair.

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Risk parity is a meaningful and robust approach for building well-diversified portfolios, but it relies on historical volatility estimates, which penalises upside risk as well as downside risk and leads to a massive overweighting of bonds versus equities, even in a low yield environment.

Risk parity has become an increasingly popular approach for building well-diversified portfolios within and across asset classes. In a nutshell, the goal of the methodology is to ensure that the contribution to the overall risk of the portfolio will be identical for all constituent assets, which stands in contrast to an equally-weighted strategy that would also recommend an equal contribution but instead simply be expressed in terms of dollar budgets as opposed to risk budgets (see Roncalli (2013b) for a formal introduction to risk parity and a detailed discussion of its applications).

While intuitively appealing and empirically attractive, this approach suffers from two major shortcomings. On the one hand, typical risk parity portfolio strategies used in an asset allocation context inevitably involve a substantial overweighting of bonds with respect to equities, which might be a problem in a low bond yield environment, with mean-reversion implying that a drop in long-term bond prices might be more likely than a further increase in bond prices.

On the other hand, standard approaches to risk parity are based on portfolio volatility as a risk measure, implying that upside risk is penalised as much as downside risk, in obvious contradiction with investors’ preferences.

This paper develops a conditional approach to risk parity, which contrasts with standard unconditional risk parity portfolios based on historical volatility estimates, with an attempt to alleviate the two aforementioned concerns. In a first step, we recognise that duration is a decreasing function of the bond yield for a coupon-paying bond. As a result, a decrease in bond yield levels should lead to an increase in bond duration, and as such, an increase in bond volatility should lead, everything else equal, to a decrease in the allocation to bonds for a risk parity portfolio. By using a robust econometric procedure that explicitly relates unobservable bond volatility levels to observable bond yield levels, we actually obtain an explicit response to the risk parity bond allocation as a function of changes in yield levels. In a second step, we define a new class of conditional risk parity (RP) portfolios with respect to downside risk measures such as semi-variance, Value-at-Risk (VaR) or expected shortfall. In all of these cases, conditional estimates of expected returns on stocks and bonds actually impact the estimated risk levels, so that an economic environment where the risk premium is historically high for stocks and low for bonds would lead to a further decrease in the allocation to bonds with respect to an unconditional risk parity technique, but also with respect to a conditional risk parity technique relying solely on volatility as a risk measure. In a last step, we analyse a competing approach that explicitly accounts for changes in risk premium levels, as opposed to having them indirectly impact the risk parity portfolio through their impact on the portfolio downside risk. From a formal perspective, this last extension is based on the recognition that risk parity is equivalent to a Sharpe ratio maximisation programme under the assumption that all asset classes...
have the same risk premium and the same correlations. While this constant risk premium assumption can be regarded as a reasonable, or at least agnostic, prior from an unconditional perspective, it is hardly defendable from a conditional perspective (that is for all possible market conditions), particularly when bond yields are low and dividend yields are high. In this context, it would be preferable to dynamically adjust the risk budgets around the long-term 50%/50% risk parity target so as to better reflect the current market environment, and we provide an analytical expression for the dynamic adjustment to risk budgets for each asset class that optimally reflects changes in market conditions, by defining these time-varying risk budgets as the set of risk budget targets that makes the risk budgeting portfolio a maximum Sharpe ratio portfolio given current estimated levels of volatility and Sharpe ratios for equities and bonds, while assuming an identical long-term Sharpe ratio for the two asset classes.

Duration-based volatility is an instantaneous and observable volatility bond measure that avoids the sample dependency and overweighting of bonds in a low interest rate environment, which inevitably follow from the use of historical volatility measures in the construction of risk parity portfolios. In response to the two major problems identified with historical volatility, namely sample dependency and backward-looking bias, we introduce in this paper an alternative bond volatility measure, which we refer to as “duration-based volatility” (in short, DUR volatility). This measure is suggested by the model-free approximation of the return on a bond portfolio as the product of the negative of duration times the yield change: thus, DUR volatility equals duration times the yield volatility. Duration is readily observable, both for a single bond and for a bond portfolio, where it is given as the weighted average of constituents’ durations, but yield volatility has to be estimated. In this sense, DUR volatility


This figure shows the yield of the Barclays US Treasury index, together with its volatility estimated by three methods: a five-year rolling window (VB-RW), a GARCH(1,1) model (VB-GARCH), and a volatility measure proportional to the duration (VB-DUR). Duration is computed using the approximation of Campbell et al. (1997).
is quasi-observable, rather than fully observable, but unlike rolling-window (RW) and the GARCH volatilities, it instantaneously reacts to changes in bond yields which translate into changes in bond duration.

As such, DUR volatility also helps to address the issue of bond over-weighting in a low-interest rate context. Indeed, duration for a coupon-paying bond can be shown to be a decreasing function of the bond yield. At the index level, where multiple constituents are involved, making the analysis less explicit, the negative relationship between duration and yield is still confirmed empirically. The consequence on bond volatility estimates over the past 30 years is shown in Exhibit 1: while RW and GARCH volatilities have been stable or even decreasing slightly, the DUR volatility has followed an increasing trend due to the decreasing trend in bond yield levels, and has eventually exceeded the former measures. Hence, at the end of our sample period, in December 2012, a risk parity portfolio of stocks and bonds has a lower allocation to bonds when DUR volatility is used instead of RW or GARCH estimates. To see that this “CRP-VOL-DUR” strategy is also more responsive to changes in market conditions than “CRP-VOL-GARCH” or “URP-VOL-RW”, one can look at the percentage of months in the sample over which bond allocation and bond yield moved in the same direction: Exhibit 2 indicates that this concordance rate is substantially higher for CRP-VOL-DUR. The results shown here refer to portfolios of stocks and bonds, but the same conclusions remain valid when we introduce commodities as a third asset class.

Risk parity portfolios constructed with a downside risk measure show a higher degree of sensitivity with respect to market conditions, and lead to further decreases in bond allocation in a low yield environment compared to risk parity portfolios constructed on the basis of volatility as a risk measure. As discussed above, the choice of volatility as the reference risk measure in the definition of risk parity portfolios can be criticised for not capturing investors’ concern over downside risk. As a result, assets with substantial downside risk can be overweighted as long as they have low volatility, which would be the case for bonds in a low yield environment, when mean-reversion back to higher yield levels would imply that most of the risk faced by investors is on the downside. A natural idea is to penalise such assets by introducing a downside risk measure in the definition of risk parity portfolios. The only mathematical requirement is that the measure should be homogeneous of degree 1 in portfolio weights, because this condition is needed to decompose the portfolio risk as a sum of contributions from constituents. Semi-volatility, defined as the volatility of negative returns, and Value-at-Risk (VaR), defined as the quantile of order of the portfolio loss distribution, both satisfy this condition. To obtain explicit expressions for these measures as functions of portfolio weights, and to compute the risk contribution of each asset and then to solve for the set of weights that equates these contributions, one needs to make an assumption on the distribution of returns. A mathematically convenient choice is the Gaussian distribution, but it is hardly appropriate for an analysis of asymmetric and fat-tailed return distributions. In this...
context, we propose to use the Cornish-Fisher expansion to obtain a correction to the Gaussian VaR, that accounts for the presence of non-zero skewness and excess kurtosis levels.

One drawback of these dissymetric risk measures is that they depend on expected returns, which are notoriously hard to estimate (see Merton (1980)). Empirical research, however, provides useful guidance with respect to the relevance of state variables such as dividend yields as proxies for the time-varying expected return on equities (see Fama and French (1988), Hodrick (1992) and Menzly et al. (2004)). Similarly, the intuition suggests that the current yield level can be taken as a predictor for bond expected returns. Indeed, the price of a bond is a decreasing function of the yield, which exhibits mean reversion. In the current environment, when bond yields have reached historically low levels, bonds appear to be expensive relative to historical standards, thus implying that their expected return is low (see Campbell et al. (1997) for a justification of this intuitive argument based on a formal approximation of bond returns). In this spirit, we construct our estimates for expected returns on stocks and bonds in two steps. First, we compute an expected return forecast from a predictive regression, the predictor being the dividend-price ratio for the stock index and the current yield for the bond index. Second, we shrink the forecast towards a prior, which is itself obtained by assuming that the risk premium on the asset equals the current volatility times the Sharpe ratio measured over a very long sample (1926-2012).

It appears from Exhibit 2 that the CRP-NGVAR99 strategy, which equates the contributions to the non-Gaussian VaR at 99%, displays a concordance rate (again defined as the percentage of months in the sample over which bond allocation and bond yield moved in the same direction) that is higher compared to other risk parity strategies, notably the one relying on the duration-based volatility measure. For Gaussian downside risk measures (not shown in the Exhibit), we find that concordance rates are between those of the Cornish-Fisher VaR and the volatility, which suggests that the highest degree of responsiveness to changes in market conditions is obtained with the risk parity portfolio strategy based on the non-Gaussian VaR risk measures. Interestingly, our results also indicate that the ratio of stock risk to bond risk tends to be substantially lower with this VaR measure compared to volatility or Gaussian risk measures, especially when bond yields are particularly low suggesting an increase in downside risk for bond portfolios. As a result, the CRP-NGVAR99 strategy tends to lead to a lower bond allocation than other risk parity strategies tested in this paper, especially by the end of the sample period when bond yield have reached unusually low levels.

The maximum Sharpe ratio portfolio, which can be interpreted as a risk parity portfolio under some conditions (identical pairwise correlations and identical Sharpe ratios), is a less robust alternative to incorporating expected returns in the construction of a well-diversified policy portfolio. Under some conditions, the maximum Sharpe ratio (MSR) portfolio can be...
interpreted as a risk parity portfolio. Indeed, Maillard et al. (2010) establish that the two portfolios coincide if all assets have the same pairwise correlations and the same Sharpe ratio. In the case of two assets, this condition reduces to the equality of Sharpe ratios, an assumption which can be regarded as a reasonable agnostic long-term prior, but which is unlikely to be satisfied in all market conditions, in particular if bond yields are extremely low and dividend yields are extremely high. Alternatively, one can also show that the MSR portfolio achieves risk parity for a risk measure defined as volatility with a penalisation for expected return, as in Roncalli (2013a).

In this paper, we test two MSR portfolios, which differ through the priors used for expected returns. Exhibit 2 shows that the concordance rates are close to those of the risk parity strategies constructed from duration-based volatility or Gaussian VaR. Hence, our results suggest that there are no additional benefits to expect from MSR strategies in this dimension. Another non-negligible drawback of these portfolios over the risk parity portfolio based on the Cornish-Fisher approximation, which uses the same expected return estimates, is that they involve a much higher level of turnover. This property reflects the well-known fact that MSR weights have a large sensitivity to expected returns, and, as a consequence, are more impacted by estimation errors in these parameters. Overall, one key difference between MSR portfolios and conditional risk parity portfolios based on downside risk measures is that even if they both rely on expected return estimates, risk parity strategies treat these estimates as (directional) risk measures as opposed to pure expected return inputs as in the case of the MSR strategy, implying a lower sensitivity to estimation risk.

Executive Summary

Conditional risk parity strategies display better performance in a scenario of increasing bond yields.

The concordance rates and bond weights give indications on the behaviour of the various risk parity strategies. In order to see the benefits of the conditional approaches in a more material way, we simulate an increase in interest rates as of December 2012 until December 2017. The use of Monte-Carlo simulations is almost an obligation here as the historical sample that was available for this study (1973-2012) is mostly characterised by a long decrease of interest rates starting in the 1980s, with no periods of sustained decreases in yields. To test for the effectiveness of improved risk parity strategies in responding to increases


<table>
<thead>
<tr>
<th></th>
<th>Concorand months (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>URP-VOL-RW</td>
<td>45.346</td>
</tr>
<tr>
<td>CRP-VOL-GARCH</td>
<td>48.087</td>
</tr>
<tr>
<td>CRP-VOL-DUR</td>
<td>67.303</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>74.582</td>
</tr>
<tr>
<td>MSR-SAME-SR</td>
<td>71.241</td>
</tr>
<tr>
<td>MSR-DIFF-SR</td>
<td>73.389</td>
</tr>
</tbody>
</table>

The concordance rate is defined as the percentage of months in the sample in which the yield and the bond weight moved in the same direction. URP-VOL-RW is the standard risk parity strategy that relies on historical volatilities; CRP-VOL-DUR is a risk parity strategy that uses duration-based volatility as the bond volatility measure; CRP-NGVAR99 is a strategy that equates the contributions of constituents to non-Gaussian VaR at 99%; MSR-SAME-SR is a maximum Sharpe ratio strategy that uses the same prior Sharpe ratio for both constituents; and MSR-DIFF-SR uses different priors.
in interest rates, we have simulated two economic scenarios: in the first one, the reversion of bond yields from their current level back to their long-term mean value is complete after five years (on average), while the second scenario assumes a more rapid increase in interest rates, with a mere two-year time period needed to reach the same value. Exhibit 3 provides an overview of the results. The main observation is that the conditional risk parity strategies perform better over the period of increases in interest rates, if only because they start with a lower bond allocation. In particular, they avoid the severe losses displayed by the URP-VOL-RW strategy in the first year of the increasing trend for interest rates. On the other hand, they also have a higher stock exposure, so they are more impacted by the performance of the stock market. To give a sense of this impact, we have assumed that in the first scenario, the equity market is not impacted, while in the second one, the rapid increase in rates results in a strong market downturn after two years. We note that the CRP-NGVAR99 strategy displays the worst performance in this year, due to a higher stock allocation, but it recovers in the next two years, where it appears to be the top performer.

Overall, our analysis suggests that it is possible to construct alternative forms of risk parity strategies that alleviate some of the concerns posed by the standard approach based on historical volatility measures. The risk parity strategy relying on duration-based volatility, which is not as dependent on past bond returns as rolling-window or GARCH volatilities, is a first step towards the introduction of an instantaneous cheapness indicator in a portfolio construction methodology that otherwise solely focuses on risk management. An improved responsiveness to changes in bond yield levels can be achieved through the use of a downside risk measure, such as the Cornish-Fisher VaR, and this increased sensitivity would result in better performance in a period of sharp increases in interest rates. While the implementation of such strategies requires

**Executive Summary**

<table>
<thead>
<tr>
<th>(a) Scenario 1 (slow increase of interest rates).</th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
<th>2016</th>
<th>2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (av.)</td>
<td>0.020</td>
<td>0.036</td>
<td>0.017</td>
<td>0.054</td>
<td>0.059</td>
</tr>
<tr>
<td>Bond index</td>
<td>-0.007</td>
<td>-0.053</td>
<td>-0.034</td>
<td>-0.015</td>
<td>-0.005</td>
</tr>
<tr>
<td>Stock index</td>
<td>0.061</td>
<td>0.062</td>
<td>0.054</td>
<td>0.059</td>
<td>0.061</td>
</tr>
<tr>
<td>URP-VOL-RW</td>
<td>-0.061</td>
<td>-0.021</td>
<td>-0.006</td>
<td>0.011</td>
<td>0.020</td>
</tr>
<tr>
<td>CRP-VOL-DUR</td>
<td>-0.048</td>
<td>-0.012</td>
<td>-0.002</td>
<td>0.013</td>
<td>0.020</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>-0.022</td>
<td>0.003</td>
<td>0.008</td>
<td>0.019</td>
<td>0.025</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b) Scenario 2 (rapid increase of interest rates).</th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
<th>2016</th>
<th>2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (av.)</td>
<td>0.031</td>
<td>0.055</td>
<td>0.063</td>
<td>0.065</td>
<td>0.066</td>
</tr>
<tr>
<td>Bond index</td>
<td>-0.184</td>
<td>-0.047</td>
<td>-0.010</td>
<td>0.007</td>
<td>0.010</td>
</tr>
<tr>
<td>Stock index</td>
<td>0.061</td>
<td>0.007</td>
<td>-0.102</td>
<td>0.036</td>
<td>0.102</td>
</tr>
<tr>
<td>URP-VOL-RW</td>
<td>-0.127</td>
<td>-0.029</td>
<td>-0.029</td>
<td>0.021</td>
<td>0.039</td>
</tr>
<tr>
<td>CRP-VOL-DUR</td>
<td>-0.110</td>
<td>-0.026</td>
<td>-0.031</td>
<td>0.021</td>
<td>0.038</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>-0.069</td>
<td>-0.018</td>
<td>-0.041</td>
<td>0.026</td>
<td>0.050</td>
</tr>
</tbody>
</table>

This table shows the simulated average bond yield (first line) and the simulated expected excess return of risk parity strategies in a period of interest rate increase. Scenario 1 corresponds to a slow increase with no impact on the equity market, while Scenario 2 is characterised by a rapid increase, and an equity bear market in 2015. Strategies are defined in the legend of Exhibit 2.
Executive Summary

an estimate for expected return parameters, these strategies show a substantially higher degree of robustness to errors in such estimates compared to standard mean-variance portfolio construction techniques. The impact of estimation errors on out-of-sample performance and the benefits of correcting weights for parameter uncertainty have been extensively studied in the literature on mean-variance optimal portfolios, but a similar work for risk parity portfolios remains to be done.
1. Introduction
1. Introduction

Risk parity has become an increasingly popular approach for portfolio construction within and across asset classes. In a nutshell, the goal of the methodology is to ensure that the contribution to the overall risk of the portfolio will be identical for all constituent assets, which stands in contrast to an equally-weighted strategy that would also recommend an equal contribution, but instead simply be expressed in terms of dollar budgets.  

Although the concept of contribution to volatility is not new in the literature (see Litterman (1996) for a definition), the term “risk parity” appears to have first been coined by Qian (2005), who shows that for a 60%/40% stock/bond portfolio, an exceedingly large fraction of the portfolio volatility can be attributed to equities, because their volatility is substantially higher compared to bond volatility. It is only if the bond allocation reaches a level of about 80% that both assets have a similar contribution to portfolio risk. Qian (2006) gave a financial interpretation of contributions to volatility by showing that under some assumptions (including notably the normality of returns), these quantities are approximately equal to the expected contributions to portfolio loss in the event of a large loss. The first formal analysis of the “equal risk contribution” portfolio was subsequently given by Maillard et al. (2010), who establish its existence and uniqueness, derive a number of analytical properties and propose numerical algorithms to compute the portfolio. An in-depth overview of the benefits and limits of risk parity portfolios, and more generally to risk budgeting portfolios which assign non necessarily equal risk budgets for constituents, is given in Roncalli (2013b), who shows that risk parity is a disciplined approach to portfolio diversification that is more meaningful than naive approaches based on dollar budgets, and more robust with respect to errors in input parameter estimates compared to standard mean-variance analysis.

While intuitively appealing and empirically attractive, risk parity suffers however from two major shortcomings. On the one hand, typical risk parity portfolio strategies used in an asset allocation context inevitably involve a substantial overweighting of bonds with respect to equities, which raises a serious concern in a low bond yield environment, with mean-reversion implying that a drop in long-term bond prices might be more likely than a further increase in bond prices (see Inker (2010) and Schachter and Thiagarajan (2011) for related arguments). On the other hand, standard approaches to risk parity are based on portfolio volatility as a risk measure, implying that upside risk is penalised as much as downside risk, in obvious contradiction with investors’ preferences. This paper can be regarded as an attempt to address the two aforementioned concerns by developing a conditional approach to risk parity based on a downside risk measure related to instantaneously observable state variables, in contrast to standard unconditional risk parity portfolios based on historical volatility estimates.

In a first step, we recognise that duration is a decreasing function of the bond yield for a coupon-paying bond. As a result, a decrease in bond yield levels should lead to an increase in bond duration, and as such, an increase in bond volatility should

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1. - More generally, one can envision to assign equal or non-equal target budgets for the contribution of each constituent to the risk of the overall portfolio. In the specific case of an equal contribution, risk parity is sometimes also known as equal risk contribution (ERC). In this paper, we use the terminology risk parity to denote a situation where the risk budgets are identical, and the terminology risk budgeting to denote the general case with potentially non-equal risk budgets.

2. - In practice, the lower performance associated with the lower allocation to equities is often compensated by the use of leverage.

3. - Equal risk contribution is the formal definition for risk parity that we adopt in this paper, but alternative definitions are sometimes given. For instance, risk parity can refer to a portfolio whose weights are inversely proportional to constituents’ volatilities, regardless of correlations. As shown by Maillard et al. (2010), this weighting scheme equates contributions only if there are only two assets or if there are three assets with identical pairwise correlations.
also lead, everything else equal, to a decrease in the allocation to bonds for a risk parity portfolio. By using a robust econometric procedure that explicitly relates unobservable bond volatility levels to observable bond yield levels, we actually obtain an explicit response to the risk parity bond allocation as a function of changes in yield levels. In a second step, we define a new class of conditional risk parity (RP) portfolios with respect to downside risk measures such as semi-variance, Value-at-Risk (VaR) or expected shortfall.\(^4\) In all of these cases, conditional estimates of expected returns on stocks and bonds actually impact the estimated risk levels, so that an economic environment where the risk premium is historically high for stocks and low for bonds would lead to a further decrease in the allocation to bonds with respect to an unconditional risk parity technique, but also with respect to a conditional risk parity technique relying solely on volatility as a risk measure. Empirical research, however, provides useful guidance with respect to the relevance of state variables such as dividend yields as proxies for the time-varying expected return on equities (see Fama and French (1988), Hodrick (1992) and Menzly et al. (2004), as well as Welch and Goyal (2008) for a survey and an evaluation of the predictive models proposed in the literature over a very long period, namely from 1871 to 2006). Similarly, the intuition suggests that the current yield level can be taken as a predictor for bond expected returns. Indeed, the price of a bond is a decreasing function of the yield, which exhibits mean reversion. In the current environment, when bond yields have reached historically low levels, bonds appear to be expensive relative to historical standards, thus implying that their expected return is low (see Campbell et al. (1997) for a justification of this intuitive argument based on a formal approximation of bond returns and Section 3 for empirical evidence that there is at least as much predictability in bond returns as what is found in equity returns). In this spirit, we construct our estimates for expected returns on stocks and bonds in two steps. First, we compute an expected return forecast from a predictive regression, the predictor being the dividend-price ratio for the stock index and the current yield for the bond index. Second, we shrink the forecast towards an agnostic prior, which is itself obtained by assuming that the risk premium on the asset equals the current volatility times the Sharpe ratio measured over a very long sample (1926-2012). Overall our results show that conditional risk parity portfolios based on downside risk measures are more responsive to changes in yields compared to standard unconditional approaches to risk parity based on historical volatility estimates, and also that they involve a lower allocation to bonds in a low yield environment, which signals an increase in downside risk on bond portfolios.

Finally, we analyse in a last step a competing approach that explicitly accounts for changes in risk premium levels, as opposed to having them indirectly impact the risk parity portfolio through their impact on the portfolio downside risk. From a formal perspective, this last extension is based on the recognition that risk parity is equivalent to a Sharpe ratio maximisation program under the assumption that all asset classes have the
1. Introduction

same Sharpe ratio and identical pairwise correlations (Maillard et al. (2010)). While this constant risk premium assumption can be regarded as a reasonable, or at least agnostic, prior from an unconditional perspective, it is hardly defendable from a conditional perspective, that is for all possible market conditions, and in particular when bond yields are low and dividend yields are high. In this context, one would like to dynamically adjust the risk budgets around the long-term 50%/50% risk parity target so as to better reflect the current market environment. We thus provide an analytical expression for the dynamic adjustment to risk budgets for each asset class that optimally reflects changes in market conditions, by defining these time-varying risk budgets as the set of risk budget targets that makes the risk budgeting portfolio a maximum Sharpe ratio portfolio given current estimated levels of volatility and Sharpe ratios for equities and bonds, while assuming an identical long-term Sharpe ratio for the two asset classes. We find a noticeable increase in turnover when comparing MSR portfolios to the risk parity portfolio based on the Cornish-Fisher approximation, which use the same expected return estimates. This property reflects the well-known fact that MSR weights have a large sensitivity to expected returns, and, as a consequence, are more impacted by estimation errors in these parameters. Overall, one key difference between MSR portfolios and conditional risk parity portfolios based on downside risk measures is that even if they both rely on expected return estimates, risk parity strategies treat these estimates as (directional) risk measures as opposed to pure expected return inputs as

in the case of the MSR strategy, implying a lower sensitivity to estimation risk (see Roncalli (2013a), who introduces a class of risk measures that are linear in expected return and volatility, for similar conclusions).5

The rest of the paper is organised as follows. In Section 2, we provide the mathematical definition of risk parity strategies, and we present the various forms of conditional risk parity strategies that will be tested in this paper. In Section 3, we turn to the implementation of the unconditional and conditional risk parity strategies in a bond-stock universe, and we analyse in particular the historical relationship between the allocation to bonds and bond yield levels. In Section 4, we assess the benefits of the conditional risk parity approaches over fixed-mix or unconditional risk parity rules in periods of increases in interest rates, before introducing commodities in Section 5. Section 6 concludes and presents suggestions for further research.

5 - The Gaussian VaR is an example of such a risk measure which is linear in expected return and volatility, with a negative loading on the former dimension and a positive loading on the latter dimension.
2. From Unconditional to Conditional Risk Parity Strategies
2. From Unconditional to Conditional Risk Parity Strategies

In this section, we provide a formal presentation of risk parity techniques, and then introduce various dynamic extensions of the standard methodology that are designed to generate a portfolio strategy more responsive to changes in market conditions.

2.1 Formal Definition of RP Strategies

We start with a reminder of the mathematical definition of risk parity (henceforth, RP) strategies. The classical reference on this is Maillard et al. (2010). We slightly deviate from it by adopting a multi-period formulation, as opposed to taking a single-period framework. The advantage of the multi-period setting is that it explicitly takes into account the possibility to rebalance. Formally, we represent uncertainty in the economy with a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \(\mathcal{F}\) being a sigma-algebra on \(\Omega\) that represents the set of measurable events, and \(\mathbb{P}\) being a probability measure on \((\Omega, \mathcal{F})\), that represents investor’s beliefs. The initial date is 0, the investment horizon is \(T\), and the intermediate dates are denoted with \(t = 1, \ldots, T - 1\). The probability space is equipped with a filtration \((\mathcal{F}_t)_{t=0}^T\), being the information set available to the investor on date \(t\). The investment universe consists of \(N\) locally risky assets, and the investor allocates funds to these assets. The rebalancing period is denoted with \(h\), and the weights imposed on date \(t\) are stacked in the \(N \times 1\) vector \(\mathbf{w}_t = (w_{1t}, \ldots, w_{Nt})\). The simple returns on the \(N\) assets over each period \([t, t + h]\) are stacked in a \(N \times 1\) vector \(\mathbf{X}_{t,t+h}\). We let \(\Sigma_t\) denote the conditional covariance matrix of the returns at the rebalancing horizon, that is \(\Sigma_t = \mathbb{V}_t[\mathbf{X}_{t,t+h}]\) (we use \(\mathbb{V}_t\) to denote the variance conditional on information available on date \(t\)). Similarly, \(\mu_t\) denotes the expected return vector, equal to \(\mathbb{E}_t[\mathbf{X}_{t,t+h}]\).

The conditional volatility of the simple portfolio return is \(\sigma_{Pt} = \sqrt{w_t^\top \Sigma_t w_t}\). Because it is homogenous of degree 1 in the weights, it admits the following decomposition into the sum of contributions to volatility (known as Euler formula):

\[
\sigma_{Pt} = \sum_{i=1}^{N} c_{it}^{vol}(\mathbf{w}_t) = w_{it} \frac{\partial \sigma_{Pt}}{\partial w_{it}}.
\]

Expliciting the partial derivative, we have, for \(i = 1, \ldots, N:\n\]
\[
c_{it}^{vol}(\mathbf{w}_t) = \frac{w_{it}}{\sigma_{Pt}} \sum_{j=1}^{N} \sigma_{ijt},
\]

where \(\sigma_{ijt}\) is the covariance between assets \(i\) and \(j\). In vector form, the vector of contributions can be written as:

\[
c_t^{vol}(\mathbf{w}_t) = \mathbf{w}_t \odot \Sigma_t \mathbf{w}_t,
\]

where \(\odot\) denotes the element-by-element product (also known as the Hadamard product).

The RP portfolio is defined by the following equalities:

\[
\begin{align*}
\{ c_{1t}^{vol}(\mathbf{w}_t) &= \cdots = c_{Nt}^{vol}(\mathbf{w}_t) \\
\mathbf{w}_t' \mathbf{1} &= 1 \\
\mathbf{w}_t &\geq 0
\end{align*}
\]

We denote its weights with \(\mathbf{w}_t^{RP,vol}\).

The existence and the uniqueness of the RP portfolio is proved in Spinu (2013). In general, no analytical expression is available for the weights, so that they have to be computed numerically. One situation where weights can be computed...

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Proposition 1 Assume that all assets have the same pairwise correlations. Then, the RP portfolio is given by:

$$w_{it} = \frac{\sigma^{-1}_{it}}{\sum_{j=1}^{N} \sigma^{-1}_{ij}}, \quad i = 1, \ldots, N.$$  

A fundamental property of the RP strategy, which follows from the equality of contributions in (2.1), is that it tends to overweight low volatility constituents with respect to high volatility ones. It also tends to overweight constituents that have low correlations with the others. It shares these properties with the global minimum variance portfolio, but unlike it, it allocates non-zero weights to all constituents. As a result, it tends to be less concentrated.\footnote{The concentration of the GMV can be reduced by imposing norm constraints on weights (see DeMiguel et al. (2009a)).}

More generally, one can define risk budgeting portfolios, in which the contributions to volatility are not necessarily equal, but are set to given values. Another extension, which we shall extensively study in this paper, is to replace volatility by a different risk measure. We will subsequently discuss the financial motivations for the choice of a different risk measure, but we simply note at this stage that if $R_t$ is a function of weights homogenous of degree 1 which is known on date $t$, then the Euler decomposition gives:

$$R_t(w_t) = \sum_{i=1}^{N} c^R_{it}(w_t),$$

$$c^R_{it}(w_t) = w_{it} \frac{\partial R_t(w_t)}{\partial w_{it}}. \quad (2.2)$$

Examples of such a risk measure is given by Roncalli (2013a), who define $R_t(w_t) = -w_t^\top \mu_t + c \sigma_{P_t}$, where $c$ is a constant. The RP portfolio is then defined by the condition

$$\left\{ \begin{array}{l} c^R_{it}(w_t) = \cdots = c^R_{Nt}(w_t) \\ w_t^\top 1 = 1 \\ w_t \geq 0 \end{array} \right. \quad (2.3)$$

To distinguish it from the volatility-RP portfolio, we use the notation $w^{R,RP}$ for its weights.

As appears from (2.1), the RP portfolio based on volatility depends on the conditional covariance matrix $\Sigma_t$, which has to be estimated at each rebalancing date (see Section 2.3).

A variety of estimation methods are available, and as we argue in the following sections, different estimators may imply different properties for the RP strategy. In this paper, we will thus consider different RP strategies, which essentially differ through the volatility estimators. On the other hand, one of the key advantages of risk parity is that it is a sophisticated approach to portfolio diversification that does not require the estimation of expected returns. Indeed, it has long been recognised that expected returns are almost impossible to estimate accurately from past
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realised returns (Merton, 1980). Given that the mean–variance efficient portfolios of Markowitz (1952) and the maximum Sharpe ratio portfolio of Sharpe (1964) are extremely sensitive to changes in expected return inputs (Best and Grauer, 1991), the presence of substantial estimation errors in these parameters can easily offset any benefits of scientific diversification with respect to the equal weighting rule (Best and Grauer, 1991). A second option is to use weight constraints as a mean to reduce the impact of estimation errors on portfolio weights: Best and Grauer (1991) impose nonnegativity constraints, and DeMiguel et al. (2009a) apply norm constraints in order to impose a minimum level of naive diversification, and show that this improves out-of-sample performance. A third option is simply to give up on expected return estimation, by using a weighting scheme that does not require such inputs. In this context, the global minimum variance portfolio is a natural candidate since it coincides with the maximum Sharpe ratio portfolio when all expected excess returns are equal. A more extreme choice, which does not require any parameter estimate, is the equally-weighted portfolio. Maillard et al. (2010) show that the risk parity portfolio represents a midpoint between these two strategies, in the sense that for a given covariance matrix its volatility is lower than the volatility of an equally-weighted portfolio and higher than the volatility of a minimum variance portfolio. Overall, risk parity appears to be a more meaningful approach to naive diversification than equal weighting because it does not ignore differences in volatilities and correlations.9

2.2 Limitations of Unconditional Risk Parity

The volatility-RP portfolio requires an estimate for $\Sigma_t$ at each rebalancing date. Traditionally, this is done by taking the sample covariance matrix of past returns over a rolling window, but this approach raises two statistical issues:

1. sample risk: the estimate depends on a particular historical scenario, and is thus not fully representative of the true distribution of returns;
2. non stationarity risk: returns are in fact not stationary, so that the rolling-window estimate corresponds to some mean of past true covariance matrices, while one is interested in the future covariance matrix, i.e. the covariance matrix of future returns conditional on information available to date.

The sample risk issue has been the focus of a large body of literature. To summarise it briefly, the sample covariance matrix is a reliable estimator of the true matrix as long as the sample size is much greater than the number of assets, $N$, and robustification techniques are needed when this condition is not satisfied. A more extreme choice, which does not require any parameter estimate, is the equally-weighted portfolio. Maillard et al. (2010) show that the risk parity portfolio represents a midpoint between these two strategies, in the sense that for a given covariance matrix its volatility is lower than the volatility of an equally-weighted portfolio and higher than the volatility of a minimum variance portfolio. Overall, risk parity appears to be a more meaningful approach to naive diversification than equal weighting because it does not ignore differences in volatilities and correlations.9

8 - The global minimum variance portfolio lies on the efficient frontier, and it is the only mean–variance efficient portfolio that does not require any expected return estimate. When expected returns are equal, the efficient frontier reduces to a single point, which is the minimum variance portfolio.
9 - In the case of two assets or in the case of any number of assets with equal pairwise correlations, risk parity weights take a very simple form, since constituents are weighted by the reciprocals of their volatilities. As a result, the strategy will allocate a higher weight to the less volatile asset class. In a portfolio with more than two assets, or with non-equal correlations, weights are more complicated to analyse due to the absence of a closed-form expression, but the risk parity portfolio still tends to favour constituents with low volatilities and low correlations with the other constituents.
10 - See Coqueret and Milhau (2014) for a survey and a comparison of these techniques.
other hand, non stationarity is a serious concern since current true values for risk parameters do not necessarily coincide with their past values. For instance, an asset class may have low historical volatility, which leads to it being overweighted in the RP portfolio, and have greater volatility after rebalancing has been done: as a result, the asset in question will have a higher-than-expected contribution to the ex-post realised variance of the portfolio.

Historical volatility estimates also pose two other related problems. First, they do not make a distinction between downside risk and upside risk. This is a problem because investors are more sensitive to downside risk. Moreover, downside risk can vary even though volatility does not substantially vary. As an illustration, Figure 1 shows that the rolling-window volatility of the bond index did not change much over the 1990-2012 period, while the yield followed a decreasing trend. To the extent that the yield has ability to forecast future returns (which we shall confirm in Section 3.3.1), the current low yields are likely to announce poor returns in the future, meaning that the downside risk of bonds has been progressively increasing over the sample – an increasing riskiness which is not reflected by forward-looking RW volatility estimates. This brings up the last problem raised by historical volatility: it has no clear relationship to any expected return estimate, which implies that it disregards any information contained in variables that have been shown to have predictive power.

We elaborate on the statistical issues associated with RW volatility in Section 2.3, and we discuss the use of downside risk measures in Section 2.4 and the introduction of expected return estimates in Section 2.5. In what follows, we refer to the rolling-window volatility estimate as the “RW volatility”, and we denote the RW volatility of asset $i$ as $\tilde{\sigma}_{i, RW}$. Because RW volatility is not explicitly related to current market conditions, we coin the term “Unconditional risk parity strategy”, which we abbreviate as URP-VOL-RW.

2.3 Conditional Risk Parity I — Using an Instantaneous Measure of Volatility

In this section, we define a first form of “Conditional risk parity” strategies, as RP strategies that make use of a volatility measure which is explicitly related to current market conditions.

2.3.1 GARCH Volatility

An option to address the stationarity issue associated with the RW volatility is to use a GARCH or M-GARCH model. GARCH models were introduced by Bollerslev (1986), and they deliver forward-looking volatility estimates by explicitly taking into account the dynamics of volatility. The M-GARCH is a multivariate extension of GARCH models, which also allows for time-dependencies in conditional correlations (see Bollerslev et al. (1988)). One can use a GARCH model if the RP portfolio is a function of volatilities only, as it is the case with two assets only or with equal correlations across assets, or if the RP portfolio is a function of both volatilities and correlations, but one does not want to model dynamic correlations. The most simple of these models is a GARCH(1,1) with constant mean for stock and bond returns: the conditional expected return is constant, and the conditional
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variance of the return depends on the lagged innovation to the asset (ARCH(1) term) and its own lagged value (GARCH(1) term). Formally, the model reads:

\[ R_{i,t+1} = K + \varepsilon_{i,t+1} \]
\[ \varepsilon_{i,t+1} = \sigma_{i,t+1} z_{i,t+1} \]
\[ \sigma_{i,t+1}^2 = \alpha_0 + \alpha_1 \sigma_{i,t}^2 + \gamma_1 (\varepsilon_{it})^2, \tag{2.4} \]

where \( z_{i,t} \) is normally distributed with zero mean and unit variance and is independent from \( \mathcal{F}_t \). The quantity \( \sigma_{i,t} \) is the volatility of the return over the period \([t, t + 1]\), conditional on the information available to date \( t \). It is more forward-looking than a rolling-window estimate because it takes into account the dynamic evolution of volatility over time and expresses next period’s volatility as a function of past volatilities and realised returns. In what follows, we refer to the volatility estimate issued from the GARCH(1,1) model as the “GARCH volatility” estimate, and we denote this estimate for asset \( i \) as \( \sigma_{GARCH,t} \).

We use the abbreviated name CRP-VOL-GARCH for an RP strategy that uses GARCH volatilities for constituents. Although it does not assume stationary returns as the RW volatility does, the GARCH volatility is also subject to sample risk, since the parameters of the GARCH model (2.4) have to be estimated from the data, and has no direct relationship with bond yield levels.

2.3.2 Duration-Based Volatility

In this section, we argue that it is possible to construct a volatility measure for the bond index that does not rely on past returns, and is directly related to bond yield levels. To see this, consider a first-order approximation of a bond index return by the means of the duration. Formally, consider a single bond with price \( C_t \) that makes the deterministic coupon payments \( c_1, ..., c_m \) on dates \( t_1, ..., t_m \) and redeems the face value \( F \) at date \( t_m \). By definition, the (continuously compounded) yield-to-redemption \( \theta_t^C \) satisfies:

\[ C_t = \sum_{i=1}^{m} h_i e^{-(t_i-t)\theta_t^C} + F e^{-(t_m-t)\theta_t^C}, \tag{2.5} \]

and the (modified) duration of the bond is defined as the negative of the sensitivity of the price with respect to the yield:

\[ D_t^C = -\frac{1}{C_t} \frac{\partial C_t}{\partial \theta_t^C}, \tag{2.6} \]

which leads to the well-known expression of the duration as the weighted sum of cash-flow maturities divided by the price:

\[ D_t^C = \frac{1}{C_t} \sum_{i=1}^{m} (t_i - t) h_i e^{-(t_i-t)\theta_t^C} \]
\[ + (t_m - t) F e^{-(t_m-t)\theta_t^C}. \]

By (2.6), the price change associated with a small change \( \Delta \theta_t \) in the yield is approximately:

\[ \frac{\Delta C_t}{C_t} \approx -D_t^C \times \Delta \theta_t. \tag{2.7} \]

This suggests to estimate the volatility of the bond as the product of the duration and the estimated yield volatility:

\[ \bar{\sigma}_{C-DUR,t} = D_t^C \times \bar{\sigma}_{\theta,t}, \tag{2.8} \]

For a bond index, the duration \( D_t \) and the yield \( \theta_t \) are defined as the sums of durations and yields of constituents, weighted by the constituent weights. Replacing the parameters (duration and yield) of the individual bond in (2.8) by those of the bond index, we can construct a bond volatility estimator as:
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This “duration-based volatility” (in short, DUR volatility) measure is by construction more responsive to changes in yield levels compared to rolling-window or GARCH volatility estimates (see Section 2.3.3). Owing to this improved responsiveness, we call an RP strategy that uses this bond volatility estimate a “Conditional risk parity” one. Because duration is instantaneously observable, duration-based volatility estimates would also be observable, except for the need to estimate yield volatility. In general, yield volatility is itself stochastically time-varying, and bond return volatility therefore appears to be a function of two state variables, namely bond duration and yield volatility. Hence, we could use bond duration as a conditioning variable to help sharpen bond return volatility estimates for a given level of bond yield volatility. In what follows, we assume for simplicity a constant bond yield volatility, which allows us to focus on changes in bond return volatility that are driven by changes in duration, which itself can be regarded as a function of bond yield levels, as is explained in the next section.

2.3.3 Relation Between Duration-Based Volatility and Bond Yield

As the DUR volatility estimate is proportional to bond duration, it is expected to inherit some of the properties of duration, including the relationship with respect to bond yield levels, which is provided in the following Proposition.

Proposition 2 (Duration of A Coupon-Paying Bond) Consider the coupon-paying bond whose price is given in (2.5), and its duration, in (2.6). We have that the duration of this bond is strictly decreasing in the yield for \( t < t_{m-1} \), and equal to \( t_{m-t} \) (hence independent from the yield) for \( t \in (t_{m-1}, t_m) \).

Proof. See Appendix A.1.

Strictly speaking, this proposition does not apply to a bond index, which is a basket of coupon bonds, but it suggests that index duration is also decreasing in the yield. Thus, we expect the DUR volatility estimate to be decreasing in bond yield levels, that is unless the time variation in the yield volatility estimate conflicts with this effect, which cannot happen in the simple setting with constant yield volatility. The fact that we are dealing with a bond index rather than a single bond may nonetheless weaken the link between duration and yield. Indeed, Proposition 2 shows that a decrease in yield will imply an increase in duration, other things being equal, but because the composition of the index is periodically revised, the coupon rates and the cash-flow schedules of the constituents are not invariant over time. In order to make sure that we use a duration measure which is strictly decreasing in the yield, it would be desirable to express duration as an explicit function of the yield. Campbell et al. (1997) establish an approximate expression for the duration of a coupon bond selling close to par (this becomes an exact equality if the bond is exactly at par):

\[
D_t \approx \frac{1 - e^{-m\theta_t}}{1 - e^{-\theta_t}},
\]

where \( m \) is the bond maturity. Differentiating the right-hand side, it is easy to check that it is a (strictly) decreasing function of the yield. In order to make a clear distinction between the observed duration and the approximate duration in the remainder of the paper, we use the notations \( D_t^{obs} \) and \( D_t^{PPP} \).
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In Section 3, we shall confirm that the observed duration indeed covaries negatively with the yield (the approximate one, by construction, has an almost perfect negative correlation with the yield: the correlation is “almost” rather than “fully” perfect because the duration is not linear in the yield). We will use the abbreviated names CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP for the two conditional risk parity strategies based on the observed or the approximate duration, we will refer to the two volatility estimates as DUR-OBS and DUR-APP volatilities, and will use the notations \( \sigma^2_{DUR-OBS,t} \) and \( \sigma^2_{DUR-APP,t} \).

2.4 Conditional Risk Parity II – Extending Risk Parity to Downside Risk Measures

One shortcoming of volatility as a risk measure is that it is symmetric. As a result, it does not recognise that in some economic environments, downside risk is more substantial than upside risk for a given asset class. In what follows, we describe RP strategies based on downside risk measures.

2.4.1 Gaussian Semi-Volatility and Value-at-Risk

A first idea is to replace volatility by semi-volatility, which is defined as the square root of the second-order lower partial moment:

\[
\text{GSV}_a^\sigma(w_t) = \sqrt{E_t \left[(w_t'X_{t,t+h} - a)^2 1\{X_{t,t+h} \leq a\}\right]},
\]

(2.11)

where \( a \) is a fixed threshold. In what follows, we set \( a = 0 \) because this is the most natural choice for defining downside risk as the risk of experiencing a negative return. Another reason, of technical nature, is that 0 is the only value for \( a \) that makes semi-volatility an homogenous function of degree 1 in the weights, which allows us to decompose it according to Euler formula (see (2.2)).

In order to compute the partial derivatives of semi-volatility with respect to the weights, one needs an explicit expression for the portfolio semi-volatility. It can be computed in closed form under the assumption that the portfolio return is normally distributed.\(^{11}\)

Thus, we define the Gaussian semi-volatility of the portfolio as the semi-volatility of a normally distributed variable with mean \( w_t'\mu_t \) and variance \( w_t'\Sigma_t w_t \). The following proposition provides an expression for the semi-variance, which is the squared semi-volatility.

Proposition 3 (Semi-variance of A Normal Distribution) Let \( Z \) be normally distributed with mean \( \mu \) and variance \( \sigma^2 \), and \( a \) be any real number. Then:

\[
E[(Z - a)^2 1\{Z \leq a\}] = \sigma^2 + (\mu - a)^2 \mathcal{N} \left( \frac{a - \mu}{\sigma} \right) - (\mu - a)\mathcal{N} \left( \frac{a - \mu}{\sigma} \right).
\]

where \( \mathcal{N} \) and \( n \) are respectively the cumulative distribution function and the probability density function of the standard normal distribution. The semi-variance is strictly decreasing with respect to \( \mu \), and is strictly increasing with respect to \( \sigma \) if \( \mu \) is positive.

Proof. See Appendix A.2.

Interpreting \( Z \) as the return \( w_t'X_{t,t+h} \), we have \( \mu = w_t'\mu_t \) and \( \sigma^2 = w_t'\Sigma_t w_t \). The Gaussian semi-volatility can thus be written as an explicit (i.e. without any expectation operator) function of \( w_t \), which allows to derive expressions for the risk contributions (see Appendix B). The RP portfolio is obtained...
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by equating these contributions. As in the case of the volatility-based portfolio, it is not possible in general to obtain the portfolio weights in closed-form, but the system (2.1) can be numerically solved. Moreover, under certain assumptions on parameter values, the computation of this portfolio can be simplified by noting that the weights are inversely proportional to the constituent expected returns or to the constituents’ risk measures (see Appendix C), so that a numerical solver is not even needed.

The proposition also shows that the portfolio semi-volatility is decreasing in the portfolio expected return. This is a desirable property, because it implies that an asset with low expected return will have high semi-volatility, hence will tend to receive a lower weight in the RP portfolio based on semi-volatility. In what follows, we denote this RP strategy as CRP-GSVOL, and we will empirically check the link between the bond weight in this strategy and the bond yield, taken as a predictor of future returns.

Another commonly used risk measure is the portfolio Value-at-Risk (VaR), which measures the size of “large” potential losses. Given a confidence threshold $\alpha$, for which typical values are 95% or 99%, we define the portfolio VaR of order $\alpha$ as the quantile of order $\alpha$ of the distribution of the loss over one rebalancing period. Mathematically, this definition reads:

$$ \text{P}_t \left( -w'_t X_{t,t+h} \leq \text{VaR}^\alpha_t \right) = \alpha. $$

(2.12)

The VaR is clearly homogenous of degree 1 in the weights, so the Euler decomposition (2.2) applies. To compute the portfolio such that all contributions to VaR are identical, one needs to have an analytical expression of VaR as a function of portfolio weights. Such an explicit expression can be obtained if one assumes that the portfolio return is normally distributed, which leads to the Gaussian VaR:

$$ \text{GVaR}^\alpha_t (w_t) = -w'_t \mu_t + \mathcal{N}^{-1}(\alpha) \sqrt{w'_t \Sigma_t w_t}, $$

(2.13)

where $\mathcal{N}^{-1}(\alpha)$ is the quantile of order $\alpha$ of the standard normal distribution. The Gaussian VaR thus fits into the class of risk measures introduced by Roncalli (2013a), all of which are decreasing functions of expected returns.

2.4.2 Non-Gaussian Value-at-Risk

The assumption of Gaussian returns is hardly appropriate for assessing the VaR of a portfolio. Indeed, the VaR relates to extreme losses, and the normal distribution has thin tails, which may lead to underestimating the frequency and the size of potential losses. A correction is often applied to Gaussian VaR to incorporate the effects of higher order moments (skewness and kurtosis). This correction leads to the Cornish-Fisher expansion of the VaR (Cornish and Fisher, 1938), to which we refer as "non-Gaussian VaR":

$$ \text{NGVaR}^\alpha_t (w_t) = \text{GVaR}^\alpha_t (w_t) $$

$$ + \left[ -\frac{q^2}{6} \text{Sk}_t(w_t) ight. $$

$$ + \left. \frac{q^3}{24} \text{Ku}_t(w_t) \right] \sigma_{Pt}, $$

$$ - \frac{2q^3}{36} \text{Sk}_t^2(w_t) \sigma_{Pt}, $$

(2.14)
where \( q = \mathcal{N}^{-1}(\alpha) \), and \( \text{Sk}_t(\mathbf{w}_t) \) and \( \text{Ku}_t(\mathbf{w}_t) \) are the portfolio skewness and kurtosis:

\[
\text{Sk}_t(\mathbf{w}_t) = \frac{\mathbb{E}_t\left[(w'_tX_{t,t+h} - \mu_{Pt})^3\right]}{\sigma_{Pt}^3},
\]

\[
\text{Ku}_t(\mathbf{w}_t) = \frac{\mathbb{E}_t\left[(w'_tX_{t,t+h} - \mu_{Pt})^4\right]}{\sigma_{Pt}^4} - 3,
\]

(2.15)

and \( \mu_{Pt} = \mathbf{w}'_{t}\mu_t \). If returns are normally distributed, then the skewness and the kurtosis cancel, and the non-Gaussian VaR agrees with the Gaussian one, as it should.

The use of non-Gaussian VaR undoubtedly leads to a more realistic assessment of the extreme loss risk than the Gaussian VaR for asset return distributions that have fat tails. Nevertheless, it requires the estimation of higher order moments of assets: not only the individual skewnesses and kurtosis are needed, but one also needs the full co-skewness and co-kurtosis matrix, in order to compute portfolio higher order moments. Since there are \( N(N+1)(N+2)/6 \) co-skewnesses and \( N(N + 1)(N + 2)(N + 3)/24 \) co-kurtosis, the number of parameters to estimate is a quickly growing function of \( N \), which raises robustness concerns in large universes.\(^{12}\) In an asset allocation context with two or three asset classes, however, this concern should be of limited importance in practice.

The RP portfolio is defined by the conditions (2.1), with volatility being replaced by the non-Gaussian VaR. The derivation of risk contributions is complicated by the presence of higher order moments of portfolio return, but it can still be carried out analytically (see Boudt et al. (2008) and Appendix B).

2.5 Conditional Risk Parity III - Deviate from Parity in the Short Run while being at Parity in the Long Run

Like the CRP strategies based on downside risk measures, the last form of RP strategies that we introduce aims at making weights explicitly depend on expected returns. A simple way of creating a dependence between weights and expected returns is to consider the maximum Sharpe ratio (MSR) portfolio, which by definition achieves the highest Sharpe ratio. Denoting the risk-free rate for the period \( [t, t + h] \) as \( r_r \), we have that the MSR portfolio solves:

\[
\max \frac{w'_t\mu_t - r_r}{\sqrt{w'_t\Sigma_{t}w_t}} \quad \text{subject to} \quad \begin{cases} w'_t\mathbf{1} = 1 \\ w_t \geq 0 \end{cases}
\]

(2.16)

We denote its weights as \( \mathbf{w}_t^{\text{MSR}} \). If there were only one period, it would be the portfolio that every investor should hold in combination with the risk-free asset in order to maximise return subject to a volatility constraint (Sharpe, 1964). For simplicity of exposure, we assume that parameter values \( (\mu_t, \Sigma_t) \) are such that (2.16) defines a unique portfolio. If this condition is not satisfied, then the results stated in Propositions 4 and 5 remain valid, by changing “the MSR portfolio” into “any MSR portfolio”.

The following proposition states conditions under which the MSR portfolio coincides with the volatility-RP portfolio. It involves the \( N \times 1 \) vector of expected excess returns, defined as \( \tilde{\mu}_t = \mu_t - r_t \mathbf{1} \), the \( N \times N \) correlation matrix \( \Omega_t \), and the \( N \times 1 \) vector of Sharpe ratios \( \Lambda_t \), the Sharpe ratio of asset \( i \) being defined as \( \lambda_{it} = \frac{\tilde{\mu}_{it}}{\sigma_{it}} \).

12 - See Martellini and Ziemann (2010) for an application of shrinkage techniques to the estimation of higher order comoments in the presence of a large number of constituents.
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**Proposition 4** Assume that all expected excess returns are nonnegative. Then the MSR portfolio (defined in (2.16)) coincides with the volatility-RP portfolio (defined in (2.1)) if, and only if, the following conditions are satisfied:

\[
\begin{align*}
\Sigma^{-1}_t \hat{\mu}_t &\geq 0, \\
\hat{\mu}_t \odot \Sigma^{-1}_t \hat{\mu}_t & = x 1
\end{align*}
\]

for some positive \(x\), \(2.17\)

two conditions that may be rewritten as:

\[
\begin{align*}
\Omega^{-1}_t \Lambda_t &\geq 0, \\
\Lambda_t \odot \Omega^{-1}_t \Lambda_t & = x 1
\end{align*}
\]

for some positive \(x\). \(2.18\)

Under these conditions, the weights of the MSR portfolio are inversely proportional to expected excess returns.

Conditions (2.17) are satisfied in particular if all assets have the same pairwise correlation \(\rho\) such that \(\rho > -1/(N-1)\), and the same positive Sharpe ratio. In this case, the MSR weights are inversely proportional to volatilities.

**Proof.** See Appendix A.3.

From Proposition 4, the MSR and the volatility-RP portfolio coincide if all correlations and all Sharpe ratios are equal. In the case of two assets, as with one stock and one bond, this condition reduces to the equality between the two Sharpe ratios. If we assume that the stock and the bond have the same long-term Sharpe ratio, then the long-term MSR portfolio (that is, the one computed with long-term parameters) coincides with the long-term volatility-RP portfolio. While the assumption of equal long-term Sharpe ratios may not be strictly validated by the empirical analysis of long-term estimates (see for example Dimson et al. (2008)), it constitutes a reasonable agnostic working assumption in the absence of any prior information on the differential performance between the two asset classes. On the other hand, it is hardly defendable from a conditional perspective, that is as a function of changing market conditions. In particular, if equity markets are unusually cheap from a historical perspective while bond markets are unusually expensive, assuming that the equity and bond Sharpe ratios are equal, which is implicitly needed to justify the RP portfolio as an MSR portfolio, can hardly be justified as a relevant agnostic prior. Thus, we do not make the assumption of equal Sharpe ratios at each date, so that the MSR portfolio does not in general coincide with the volatility-RP portfolio. The third form of CRP strategy that we introduce is precisely the MSR portfolio, defined in (2.16). A formal justification for the denomination “risk parity” is that this portfolio equates the contributions to a special risk measure, as shown in the following proposition.

**Proposition 5** Let \(\lambda_{MSR,t}\) be the Sharpe ratio of the MSR portfolio, defined in (2.16), and define the risk measure

\[
R_t(w_t) = -w_t^t \hat{\mu}_t + \lambda_{MSR,t} \sqrt{w_t^t \Sigma_t w_t}.
\]

Then, the MSR portfolio achieves risk parity with respect to this measure.

**Proof.** See Appendix A.4.
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In this section, we describe the empirical implementation of the various CRP strategies described in the previous section. We do this in a two-asset framework, where the two constituents are a stock index and a bond index.

3.1 Data Base
The stock index is represented by the CRSP value-weighted index of the S&P 500 universe, with dividends reinvested, available from January 1927 through December 2012 at the daily frequency. Price index and dividends are also extracted. The bond index is represented by the Barclays US Treasury bond index: it consists of US Treasury bonds with fixed coupon payments (no embedded options), having a minimal time-to-maturity of one year. A minimum amount outstanding and a minimum rating are also required for inclusion of a bond in the index. Finally, the index is rebalanced on a monthly basis. We use the total return since inception, which reflects the gains from reinvesting the cash flows in the index. We also download timeseries information on the duration and the redemption yield. The history for this index is much more limited than that of the stock, since it starts only in January 1973, and has only monthly observations until March 1994, where daily quotes start to be available. As the short-term interest rate, we take the 3-month Treasury bill Secondary market rate, available from the Federal Reserve website at the daily frequency as of 1954. The coverage period and the frequency of the data used in our study are constrained by the availability of the bond index, so we conduct our empirical analysis on monthly data over the 1973-2012 period. We compute the dividend-price (DP) ratio of the stock as the sum of dividends paid in the previous 12 months, divided by the current price index:

\[ \text{DP}_t = \frac{\sum_{k=0}^{11} d_{t-k}}{S_t}, \]  

(3.1)

where \( S_t \) denotes the price index, and \( d_t \) is the dividend paid in period \([t, t+1]\).

Table 1 contains summary statistics on the various series. The expected return on the bond index (7.6% per year) looks overstated, which implies a suspiciously high Sharpe ratio of 0.44, because it is obtained on a sample that contains the longest bull bond market in history. We also note that the sample volatility of bond returns is about one third that of stock returns; hence an RP portfolio that uses historical volatilities as risk measures will overweight the bond index with respect to the stock index by a factor of 3 to 1. Negative skewnesses and positive kurtosis (in excess of the kurtosis of the normal distribution) for stock and bond returns are classical indications that point to the non-normality of returns. Finally, the correlation matrix shows several noteworthy properties of bond returns. First, they have close to −1 correlation with yield changes, which is natural since the yield is a decreasing function of the price. The correlation, however, is not exactly −1 because the relation between the yield change and the return is not strictly linear, and because other parameters of the bond index (coupon rates and cash flow schedules of constituents) vary from one date to the other. Besides, there is also a strong negative correlation between bond yield and bond duration, both in level (−0.718) and in difference (−0.461). This provides an empirical validation of Proposition 2 in the context of a bond index. Finally, the table

13 More formally, one can use a Jarque-Bera test, whose statistics is a function of the sample skewness and kurtosis, in order to test for the normality of returns.
also shows that stock returns are strongly negatively correlated with changes in the DP ratio; this property is in fact a natural consequence of definition (3.1), since the DP ratio contains the price index as the denominator.

3.2 CRP I Strategies

This section describes the implementation of the URP-VOL-RW strategy, as well as the two forms of CRP strategies based on volatility.

3.2.1 Estimating Bond and Stock Volatilities

The RW volatilities of the stock and the bond are the sample volatilities of returns computed over a five-year rolling window. The length of five years is chosen in order to have a large number of datapoints included in the sample (60 observations, since we use daily returns). The GARCH volatilities are obtained by estimating the GARCH(1,1) model (2.4) over a five-year rolling window, and the GARCH volatility on date \( t \) is the estimate for the conditional volatility of the return over the period \( [t, t + 1] \). The model is estimated over a rolling window as opposed to the entire period, in order to avoid a look-ahead bias. The DUR-OBS volatility estimator is computed as (2.9) with duration being equal to the observed duration, and the yield volatility taken to be constant: this constant is equal to the sample annualised standard deviation of monthly changes in the index yield, that is 1.48%. The DUR-APP estimator is computed with the same yield volatility, but duration is approximated as (2.10): the maturity \( m \) that appears in this formula is chosen in order to match the averages of approximate and observed durations over the 1973-2012 period. This condition leads to \( m = 5.37 \) years.

Figure 1 shows the various volatilities, expressed in annual terms, and Table 2 provides descriptive statistics on the various estimators. For both asset classes, the rolling-window (RW) and the GARCH estimators give close average volatilities: 5.3% and 5.4% respectively for the bond index, and 15.2% and 15.7% for the stock index. On the other hand, for the bond index, the two duration-based estimators have higher average values: 7.2% for the DUR-OBS and 7.0% for the DUR-APP estimators. Figure 1 shows that these higher average values come from higher volatility values as of the middle of the 1980s. After a high regime, above 6%, between April 1980 and October 1983, and a peak at 8.72% in 1984 for the RW estimate, the RW and the GARCH estimates went down to lower levels for a long period. They subsequently stayed below the two duration-based estimators for the remainder of the period.

The duration-based estimators followed a different path, with relatively low levels, below 7%, in the beginning of the sample, and an upwards trend from 1984 onwards. For the DUR-OBS estimator, this is explained by the fact that the Barclays index duration has been increasing over time, starting from approximately 3 years at the inception date, to exceed 5 years as of 1997 and reach 6 years in the early 2000s. For the DUR-APP estimator, the increasing trend is explained by the decreasing trend of the bond yield index, and the fact that the approximate duration is by construction a decreasing function of the yield.

The correlations between volatility measures in Table 2 point to the existence of two
3. Implementation of CRP Strategies

groups. On the one hand, the RW and GARCH volatilities have a strong correlation of 60.7%, and have close dispersions. On the other hand, the DUR-OBS and DUR-APP volatilities have lower volatilities, and have an even stronger correlation of 83.6%. The correlations between volatilities of the two groups are all negative, which reflects their diverging behaviours in the time series (Figure 1). For the stock index, the RW and GARCH volatilities are also positively correlated, which confirms that these two estimation methods do not give fundamentally different results.

As these volatility measures are to be used for the construction of RP strategies, and since we are concerned with the relation between bond weight and the yield, it is interesting to analyse the link between bond volatility for each methodology and bond yield levels. We report in Panel (a) of Table 3 the correlations between volatility and yield changes, but because the relation between these two variables is not necessarily linear, we also report another indicator, which is the “discordance rate”: it is the percentage of months in the sample in which the volatility and the yield moved in opposite directions (the change over a given month is measured between the last calendar day of the previous month and that of the current month). The reason for focusing on the discordance rate is because the bond weight in the RP portfolio will tend to be decreasing in the risk measure: hence, for the weight to be small when the yield is small, we must have a negative link between the risk measure and the yield. The results show that the RW and the GARCH estimates are positively correlated with the bond yield, and have a discordance rate less than 50%, which is consistent. Hence, these estimates tend to be low when the yield is low, so that we expect an RP portfolio based on these estimates to overweight the bond index precisely at times the yield is low. Nevertheless, the discordance rates close to 50% suggest that this link is not particularly strong: it would thus be fairer to say that these estimates do not have a well-defined response to yield changes. In contrast, the two duration-based volatilities have a stronger link to bond yields, with correlations lower than −80%, and discordance rates much higher than 50% (of course, the rate is 100% for the DUR-APP volatility). Because the negative link between the DUR-APP estimator and the yield is contained in the definition of this estimator and because there are theoretical model-free reasons for a negative link between the DUR-OBS volatility and the yield (see Proposition 2), these numbers are not simply statistical artifacts and they confirm that (i) these two estimators are much more responsive to yield changes than the RW and the GARCH ones, and (ii) the response goes in the desired direction.

3.2.2 Weights of RP Strategies

In a two-asset universe, the weights of the volatility-RP strategy are simply inversely proportional to the constituent volatilities (see Proposition 1). This observation facilitates the computation of RP weights, which are shown in Figure 2. Strategies URP-VOL-RW, CRP-VOL-GARCH, CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP have a common property, which is that they are dominated by the bond, which comes as no surprise because the bond as higher volatility than the stock. But the bond weight is larger in strategies URP-VOL-RW and CRP-VOL-GARCH from 1989 onwards.

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To give a quantitative sense of the difference, the figure also gives the bond weights on the last date in the sample, which is 31 December 2012. The URP-VOL-RW and CRP-VOL-GARCH strategies have close bond weights, of 79.8% and 82.9%. Hence, the use of GARCH volatilities does not substantially modify the allocation with respect to a strategy based on RW volatilities, although the weights are now functions of market conditions. The picture changes if one replaces the RW volatility for the bond by a duration-based estimate, either the DUR-OBS or the DUR-APP volatility (stock volatility being taken equal to the RW estimate): the bond weight at the last date falls down to 70.9% and 71.2% respectively, which represents a substantial decrease with respect to the 79.8% of the URP-VOL-RW rule. This decrease was to be expected in view of Figure 1, which showed that since the mid 1980s, the duration-based volatilities have been exceeding the RW and the GARCH ones. In other words, taking a volatility proportional to the duration leads to bonds being regarded as more risky in periods of low interest rates than what a sample-based volatility estimate would suggest, and this higher perceived risk implies a decrease in the allocation.

Another indicator reflecting the difference in the behaviour of the strategies with respect to bond yield is the concordance rate between the bond weight and the yield. This quantity measures the percentage of months in the sample where the weight and the yield moved in the same direction. A high concordance rate is something desirable for the following reason. To the extent that a high yield is an advanced indicator of high bond returns (which we shall confirm in Section 3.3.1), one should seek to increase the bond allocation at times the yield goes up: this is exactly what is meant by a high concordance rate. In this perspective, the percentage of 50% is a natural reference point: if the rate falls below 50%, then the strategy tends to increase the bond allocation when the bond gets more expensive (i.e. the yield goes down), which sounds unreasonable. Panel (b) in Table 3 reports this indicator, together with the linear correlation. It turns out that strategies URP-VOL-RW and CRP-VOL-GARCH imply bond weights that covary negatively with the yield, and concordance rates that are close to 50%. Hence, the bond weight is more likely to vary in the opposed direction of the yield. On the other hand, the other two strategies display concordance rates above 50% and positive correlations, which means that bond weight tends to move in the same direction as the yield. Hence, these strategies tend to increase the allocation to the bond when it is relatively inexpensive, and tend to decrease it if the low yield makes it appear to be expensive.

3.3 CRP II Strategies

We now turn to the implementation of CRP II strategies. The main difference with respect to CRP I strategies is that they require expected return estimates, in addition to the volatility estimates already obtained.

3.3.1 Estimating Bond and Stock Expected Returns

The downside risk measures introduced in Section 2.4 depend on expected returns. Since the sample mean of historical returns is known to be a poor estimator
of these parameters (Merton, 1980), we take a standard approach which is to relate expected returns to observable variables.

**Predicting Stock Returns** As explained in the introduction, we follow Fama and French (1988) in regressing the total return on the stock on the dividend-price ratio:

$$\ln \frac{S_{t+h}}{S_t} + \sum_{k=1}^{h} d_{t+k} = \alpha_S + \beta_S D_P + \varepsilon_{S,t+1}.$$  

The “forecast” for future returns is then:

$$\hat{\mu}_{St}^{\text{forecast}} = \hat{\alpha}_S + \hat{\beta}_S D_P.$$  \hspace{1cm} (3.2)

Regression diagnostics are reported in Table 4. Letting the prediction horizon vary from one to ten years, we observe that the $R^2$ is increasing in the horizon. The estimated beta is rather stable across horizons, but its significance increases substantially. In theory, we should use the prediction horizon that coincides with the rebalancing period of our strategies, $h$, because the definition of the risk measures in Section 2.4 involves the returns over a period of length $h$. We plan to rebalance our portfolios on a quarterly basis. However, the $R^2$ of the predictive regression is only 2.3% at three months, and 4.8% at one semester. Since the $R^2$ at one year is 9.2%, we select this predictive regression, as the one that offers the best compromise between a decent fit and a horizon close to the rebalancing period.

Nevertheless, there remains a substantial uncertainty over the true value of the expected return, because the predictor explains only 9.2% of the variance of the returns. Moreover, the values of the regression coefficients $\alpha_S$ and $\beta_S$ are not known with certainty. As a consequence, we do not want to take the forecasted return (3.2) at face value, and we shrink it towards a prior. The application of shrinkage techniques to the estimation of expected returns has a long tradition (see e.g. Barry (1974), Jorion (1985) and DeMiguel et al. (2009b)). To construct the prior, we start from the long-term excess return and volatility of US large-capitalisation stocks reported by Ibbotson (2013), which relate to the 1926-2012 period. They imply a long-term Sharpe ratio $\bar{\lambda}_S = 0.41$. The prior is then obtained as the expected return that, combined with the RW volatility, implies a Sharpe ratio of 0.41:

$$\hat{\mu}_{St}^{\text{prior}} = r_t + \bar{\lambda}_S \bar{\sigma}_S - RW,t.$$  

The posterior estimator for expected return is eventually derived as a convex combination of the forecast and the prior:

$$\hat{\mu}_{St}^{\text{posterior}} = \omega_S \hat{\mu}_{St}^{\text{prior}} + (1 - \omega_S) \hat{\mu}_{St}^{\text{forecast}}.$$  \hspace{1cm} (3.3)

The last step of the procedure is to choose a shrinkage intensity: from a qualitative standpoint, the weight assigned to an estimator should be an increasing function of the confidence in this estimator. Hence, a natural idea is to estimate the respective variances of both estimators, respectively as $s_S^{\text{forecast}}$ and $s_S^{\text{prior}}$, and to set:

$$\omega_S = \frac{s_S^{\text{forecast}}}{s_S^{\text{prior}} + s_S^{\text{forecast}}}.$$  

Note that with this choice, the resulting posterior estimator has the same form as the posterior estimator of market beta established by Vasicek (1973). Because Ibbotson’s estimate for expected excess return is a sample mean, we estimate its variance by dividing the long-term variance...
of excess returns, $\bar{\sigma}_S^2$, by the number of years contained in his sample, $n_{\text{obs}}$, which leads to:

$$s_{\text{prior}}^2 = \frac{\bar{\sigma}_S^2}{n_{\text{obs}}} = \frac{0.202^2}{87} \approx 0.000469.$$  

To estimate the variance of the forecast, we first compute the variance of $\mu_{St}^{\text{forecast}}$, taking into account the uncertainty over the estimated parameters $\hat{\alpha}_S$ and $\hat{\beta}_S$. Letting $\hat{\Theta}$ be the estimated covariance matrix of these two estimators, we estimate the variance of the forecast on date $t$ as:

$$s_{St}^{\text{forecast}} = \left(1 - \text{DP}_t\right) \hat{\Theta} \left(1 - \text{DP}_t\right).$$

Finally, we estimate the variance of the forecast as the average of $s_{St}^{\text{forecast}}$ across all observations in the sample, which results in a shrinkage intensity $\omega_S = 0.57$.

Panel (a) in Figure 3 graphs the various estimates for the time-variation in expected return on equities. The forecast exhibits more variability than the prior, taking a wider range of values, and the posterior has a variance comprised between those of the two estimators.

**Predicting Bond Returns** There exists a vast literature on bond return predictability. As a starting point, the "pure expectation hypothesis", which is one of the earliest theories of the term structure, implies that there are no premiums for holding long-maturity bonds over holding short-term bills.\(^{16}\) Hence, the expected excess return on a pure discount bond over one period should simply be zero, regardless of the bond maturity. Empirical work by Fama (1984) and Fama and Bliss (1987) has provided evidence against this hypothesis by showing that the forward-spot spread has predictive power for the excess return on zero-coupon bonds, which implies that the expected excess return is time-varying and is in general non-zero. In Fama and Bliss’ work, each bond return is predicted by a different variable, because the forward rate depends on the maturity of the bond. Cochrane and Piazzesi (2005) show that it is in fact not necessary to use as many predictors as maturities, because an even higher explanatory power can be achieved with a single predictor, which is a combination of forward rates: they report $R^2$ ranging from 31% to 37%, while those of Fama and Bliss are between 5% and 14%. Strictly speaking, these findings apply to pure discount bonds, while the portfolio strategies that we consider in this paper use a bond index, which is a portfolio of Treasury coupon bonds. Several papers have studied the prediction of Treasury bonds, including Ilmanen (1995), Ilmanen (1997), Baker et al. (2003), Duffee (2011) and Joslin et al. (2013). All of them use multivariate prediction models, but because we use a single predictor, namely the dividend price ratio, for the stock, we favour a univariate approach here. The choice that would be the most consistent with Fama and Bliss (1987) would be a forward rate, but their theoretical argument for the use of the forward rate as a predictor of zero-coupon bond returns cannot be transported to a portfolio of coupon bonds. In this context, we use the yield-to-maturity (in short, yield) of the bond index (which is defined as the sum of constituents’ yields-to-maturity weighted by their weights in the index). Intuitively, the choice of this predictor makes sense and agrees with the aforementioned criticisms on the overweighting of bonds: the price of a coupon bond is a decreasing

\(^{16}\) There is in fact a variety of forms of the expectation hypothesis; see Cox et al. (1981).
function of its yield, and it is admitted that interest rates tend to revert to a positive long-term mean.\(^{17}\) Hence, low yields are likely to be followed by high yields, that is, by low bond prices, and high current yields are advanced indicators of higher future prices. In other words, the yield is a natural predictor of bond returns in the sense that it is a “cheapness” indicator of the bond relative to historical standards. Beyond the above qualitative argument, the use of this variable can be justified by the approximation of bond returns derived in Campbell et al. (1997).

More formally, Fama and Bliss (1987) show that the forward-spot spread has forecasting ability for excess returns on pure discount bonds. The use of this variable can be justified by a mathematical argument: if \(B(t, t + \tau)\) is the price on date \(t\) of a zero-coupon bond maturing at date \(t + \tau\), \(y(t, \tau)\) is the zero-coupon yield of maturity \(\tau\) and \(f(t, t + \tau - 1, 1)\) is the forward rate set on date \(t\) for the period \([t, t + \tau]\), we have:

\[
\begin{align*}
\mathbb{E}_t \left[ \ln \frac{B(t + 1, t + \tau)}{B(t, t + \tau)} \right] - y(t, 1) & = f(t, t + \tau - 1, 1) \\
& - y(t, 1) \\
& - (\tau - 1) \mathbb{E}_t [y(t + 1, \tau)] \\
& - y(t, \tau) .
\end{align*}
\]

(3.4)

Fama and Bliss argue that yields are close to a random walk, so that the expected change in the \(\tau\)-period rate is close to a constant, and the expected excess return on the bond is close to an affine function of the forward-spot spread. In order to obtain a counterpart relation, we use the approximation of Campbell et al. (1997) to the one-period return on a coupon-paying bond. Taking the notations of Section 2.3.2, we have:

\[
\ln \frac{C_{t+1}}{C_t} \approx -(D_t^C - 1)(\theta_t^{C_1} - \theta_t^C) + \theta_t^C ,
\]

(3.5)

an approximation that becomes an equality for a bond selling at par. The right-hand side consists of two terms: the first one has the same sign as the negative of yield change, and can thus be positive or negative, while the second one, the income term, which arises because of the coupon payments, is always positive. Over a single period, the yield change term is dominant, because it is scaled by the duration, which is typically much larger than 1. But over a large number of periods, the cumulative sum of income terms gains quantitative importance.

Taking expectations on both sides of (3.5) and assuming that the yield is close to a random walk (so that the expected yield change is close to a constant \(K\)), we obtain:

\[
\mathbb{E}_t \left[ \ln \frac{C_{t+1}}{C_t} \right] \approx -(D_t^C - 1)K + \theta_t^C .
\]

(3.6)

This approximation is consistent with (3.4): indeed, if one assumes that the zero-coupon curve on date \(t\) is flat, one has \(f(t, t + \tau - 1, 1) = y(t, \tau)\), so that the expected zero-coupon return reduces to an affine function of the zero-coupon rate of maturity \(\tau\). (3.6) is merely an adaptation to the case of a coupon bond.

This analysis provides motivation for the construction of our forecast for index bond returns as:

\[
\hat{\mu}_{Bt}^{\text{forecast}} = \hat{\alpha}_B + \hat{\beta}_B \theta_t ,
\]

where \(\hat{\alpha}_B\) and \(\hat{\beta}_B\) are the estimated coefficients in the regression:

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\[ \ln \left( \frac{B_{t+h}}{B_t} + \sum_{k=1}^{h} g_{t+k} \right) = \alpha_B + \beta_B \theta_t + \varepsilon_{Bt}, \]

and where \( g_t \) is the sum of coupons paid by the index constituents in period \([t, t+1]\) and \( B_t \) is the price index. Table 4 shows the results of these predictive regressions, for horizons growing from three months to 10 years. As for the stock index, we note that the \( R^2 \) tends to increase with the prediction horizon, but the effect here is not monotonic, the \( R^2 \) peaking around six years.

In order to have the same horizon as for the stock, we select the one-year regression, which gives a value of \( R^2 \) of 31.7%. The fact that \( \hat{\beta}_B \) is positive and statistically significant indicates that the yield has forecasting ability, which quantitatively confirms the straightforward intuition that expected return is increasing in the yield.

Robustification of this forecast is useful for the bond index as it was for the stock index. Even if the explanatory power of the yield with respect to bond returns is higher than that of the dividend-price ratio with respect to stock returns, and the estimated slope \( \hat{\beta}_B \) is more statistically significant than \( \hat{\beta}_S \), the \( R^2 \) is after all only of 31.8%, which leaves much of the variance unexplained.

We proceed as for the stock, by shrinking the forecast towards a prior built from the long-term Sharpe ratio of Government bonds, as implied by the values of Ibbotson (2013). We find a long-term Sharpe ratio \( \lambda_B = 0.26 \), and derive the prior by taking the DUR-APP volatility as the bond volatility:

\[ \mu_{Bt}^{\text{prior}} = \tau_t + \lambda_B \hat{\sigma}_B \cdot \text{DUR-APP}_t. \]

The posterior estimator combines the forecast and the prior, following the same rule as for the stock (see (3.3)), with a shrinkage intensity \( \omega_B \) computed according to the same protocol. That is, we take

\[ \omega_B = \frac{s_B^{\text{forecast}}}{(s_B^{\text{forecast}} + s_B^{\text{prior}})}, \]

where the prior variance is estimated as \( \hat{\sigma}_B^2 / n_{\text{obs}} \) with \( \hat{\sigma}_B = 0.097 \) and \( n_{\text{obs}} = 87 \), and the forecast variance is estimated by averaging the variances of fitted values, as was done for the stock. This results in \( \omega_B = 0.36 \).

We note that this intensity is lower than \( \omega_S = 0.57 \), which implies a higher loading on the forecast in the case of the bond than in the case of the stock. In other words, the relative uncertainty of the forecast with respect to that of the prior is greater for the stock than for the bond, which means a higher confidence level in the bond return forecast.

It should be emphasised that the expected return on the bond index does not deterministically increase in the yield because the shrinkage of the forecasted return (which is an explicit increasing function of the yield) is shrunk towards the prior (3.7), in which two effects compete: the DUR-APP volatility is a decreasing function of the yield, while the short-term rate is strongly positively correlated with the yield. But overall, we expect the posterior estimate to inherit the positive links between the forecast, the short-term rate and the yield. As a matter of fact, Figure 3 shows that the three bond expected return estimates (i.e. the forecast, the prior and the posterior) follow a decreasing pattern which is strongly reminiscent to that followed by the yield (see Panel (b) in Figure 1).

3.3.2 Downside Risk Measures for Stocks and Bonds

Having estimated volatilities and expected returns, we can compute the Gaussian
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semi-volatility and VaR of stocks and bonds, according to the expressions of Proposition 3 and Equation (2.13). It should be noted that these quantities are not of immediate interest for the computation of RP portfolios because the weights are obtained by solving (2.3), and are not merely functions of the constituents’ risk measures. This is in contrast with volatility, since the volatility-RP portfolio of two assets has weights inversely proportional to the constituents’ volatilities. For both assets, we need to make a choice between different volatilities and expected returns. It is natural to choose the posterior estimates for expected returns, because they depend on the variables that have been identified as predictors of returns, while mitigating the statistical noise that arises from the use of a limited sample by shrinking towards a long-term value. As stock volatility, we retain the RW volatility, which is in general close to GARCH volatility (see Figure 1). For bond volatility, it is in line with the objectives of this paper to choose a duration-based measure, since DUR volatilities are negatively related to the yield: this link is a statistical one for DUR-OBS volatility, and a perfect one for DUR-APP. Thus, we select the DUR-APP volatility. We compute the risk measures using monthly volatilities and expected returns, and do not annualise them since there is no clear way of annualising a monthly semi-volatility or VaR. For the two VaRs, we take a confidence level $\alpha = 99\%$.

For the non-Gaussian VaR, we also need to estimate the skewnesses and the kurtosis of the stock and the bond. The sample skewness is a poor estimate of the true value because skewness is an odd-order moment. For robustification purposes, we thus set the skewness to zero. As a matter of fact, the empirical skewnesses of stock and bond returns (reported in Table 1) are not far from zero. The stock kurtosis is estimated as the sample moment over the 1973–2012 period, that is 2.60. Bond kurtosis can be estimated in the same way, but we can also keep the idea of linking moments of the bond to those of the yield, which was done for volatility. Hence, we assume that bond returns can be accurately approximated as:

$$\frac{\Delta B_t}{B_t} \approx -D_t^{app} \times \Delta \theta_t,$$  \hspace{1cm} (3.8)

which is the transposition of (2.7) (which holds for a single bond) to a bond index. As duration is known at the beginning of the period, it can be factored out of the conditional expectation in (2.15), which leads to the following kurtosis for the bond:

$$K_t^{B} = \mathbb{E}_t \left[ \frac{(\Delta \theta_t - \mu_{\theta t})^4}{\sigma_{\theta t}^4} \right] - 3,$$

$\mu_{\theta t}$ and $\sigma_{\theta t}$ being the mean and the standard deviation of yield changes. Hence, bond kurtosis is approximately equal to yield change kurtosis, which we estimated as its sample counterpart. Eventually, the non-Gaussian VaR of the bond is computed with a constant kurtosis of 9.90.

Figure 4 shows the various risk measures. Overall, they have patterns close to those of volatility measures (see Figure 1), but there are sizable differences in terms of their respective levels. More interestingly, the ratio of estimated risk levels for stocks versus bonds varies substantially across the different measures. Indeed, the average ratio of stock volatility to bond DUR-APP volatility is 2.18, while the average ratios of stock risk measure to bond risk measure...
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for Gaussian semi-volatility and VaR and non-Gaussian VaR are respectively 2.44, 2.32 and 1.38. The conclusion is that the use of Gaussian downside risk measures does not substantially modify the relative riskiness of the two assets with respect to the use of volatility, while the non-Gaussian VaR makes the two risks comparable in magnitude. It can even be noticed that in some periods, the non-Gaussian VaR of the bond index can be greater than that of the stock index, so that the bond index appears as the more risky asset class of the two. This property can be attributed to the fact that the term within brackets in (2.14) is positive given our parameter values: hence, the non-Gaussian VaR is more sensitive to volatility changes than the Gaussian VaR (for which the term within brackets would be zero). Because DUR-APP volatility increases as the yield decreases, the non-Gaussian VaR increases more than the Gaussian VaR. Since RP weights are not simply proportional to the reciprocals of the constituents’ risk measures, one cannot draw immediate conclusions from this observation for the relative allocation to the stock and the bond in the RP portfolio. Having said this, one can expect that having closer levels for the two risk measures will avoid the large domination of bonds which is typical in RP portfolios constructed from historical volatilities.

In order to have a more formal assessment of the link between the bond risk measures and the yield, we compute the correlations between these variables, and the discordance rates, which are defined as the percentage of months in the sample where the weight and the yield moved in opposite directions. The numbers, shown in Panel (a) of Table 3, are striking, especially when they are compared to the results obtained with RW, GARCH and DUR-OBS volatilities (Panel (a) of Table 3): the correlations are very close to −1, and the discordance rates are almost at 100%. This clearly indicates that the risk measures tend to increase precisely when the yield is low. The improvement in correlation and discordance rates with respect to the three aforementioned volatilities can be attributed to two factors that operate in the same direction when the yield goes down, as it is the case in the sample. First, these risk measures are all increasing functions of volatility: for Gaussian semi-volatility, this results from Proposition 3; for Gaussian VaR, this is contained in the definition (2.13); and for the non-Gaussian VaR, this results from the definition (2.14) and our choice of parameter values, which implies that the coefficient of volatility in bond VaR is 4.64. Since the DUR-APP volatility is by construction a decreasing function of the yield, a decrease in the yield induces an increase in these risk measures through the volatility channel: this link is a deterministic one, while it would only be a statistical one if the DUR-OBS volatility was used in place of the DUR-APP one. Second, all of these measures are decreasing functions of expected return, which again follows from Proposition 3 for the Gaussian semi-volatility, and from (2.13) and (2.14) for the two versions of VaR. Because the expected return tends itself to increase with the yield, this impact reinforces the effect that flows through the DUR-APP volatility, and makes the risk measure highly responsive to yield changes.

3.3.3 Derivation of Portfolio Weights for RP Strategies

The weights of RP strategies are obtained by solving the system (2.3) at each date. For the
NGVaR-RP strategy, we use the Kronecker product representation of higher order moments, as in Jondeau and Rockinger (2006). Introducing the co-skewness and the co-kurtosis matrices

\[ M_{3t} = E_t \left[ (X_{t,t+h} - \mu_t) (X_{t,t+h} - \mu_t)' \otimes (X_{t,t+h} - \mu_t) \right], \]

\[ M_{4t} = E_t \left[ (X_{t,t+h} - \mu_t) (X_{t,t+h} - \mu_t)' \otimes (X_{t,t+h} - \mu_t)' \otimes (X_{t,t+h} - \mu_t)' \right], \]

we can rewrite portfolio third and fourth moments as:

\[ \text{Sk}_t(w_t) = \frac{w_t' M_{3t}(w_t \otimes w_t)}{\sigma_t^3}, \]

\[ \text{K}_t(w_t) = \frac{w_t' M_{4t}(w_t \otimes w_t \otimes w_t)}{\sigma_t^4}. \]

The co-skewness matrix is taken equal to zero so as to avoid estimating odd-order moments. The co-kurtosis matrix of the bond and the stock is derived from the co-kurtosis matrix of the stock and the yield changes, following the approximation (3.8). Each fourth-order comoment of the stock and the bond can thus be approximated as:

\[ E_t \left[ \tilde{r}_{S,t,t+h}^{i} \tilde{r}_{B,t,t+h}^{j} \right] \approx (-D_t^{\text{app}})^j E_t \left[ \tilde{r}_{S,t,t+h}^{i} (\Delta \theta_t - \mu_{\theta t})^{j} \right] ; \]

for any choice of integer and nonnegative indices i and j such that i + j = 4. In this equation, \( \tilde{r}_{S,t,t+h} \) and \( \tilde{r}_{B,t,t+h} \) denote the simple and centred returns on the stock and the bond. We are thus back to the estimation of the co-kurtosis matrix of the stock and yield changes.

Figure 5 shows the time-series of weights of the three RP strategies. The weights of the two strategies that rely on Gaussian risk measures (semi-volatility or VaR) are visually very close to each other, and very close to the weights of the CRP-VOL-DUR-APP portfolio (Panel (d) in Figure 2). The similarity between these weights in levels is to be related to the observation that the relative degrees of riskiness of the stock index and the bond index are roughly the same under volatility and under the two Gaussian risk measures. Regarding changes in weights, we have already noticed (Figure 4) that the movements of the downside risk measures are similar to those of volatilities, a property that the weights inherit. The non-Gaussian VaR stands out among the three measures, because it implies that bond and stock indices have less dissimilar levels of risk: it is therefore not surprising to obtain a more balanced allocation between the two constituents. As is known, an equally-weighted portfolio is far from risk parity as long as volatility is chosen as the risk measure, but Panel (c) here shows that when the non-Gaussian VaR is taken as a substitute for volatility, the RP portfolio is closer to the equally-weighted. This allocation is however very different from a fixed-mix 50%/50% portfolio since the bond allocation exhibits a decreasing trend over the period. Even if the bond allocation is not simply proportional to the reciprocal of the non-Gaussian bond VaR, this decrease in the weight can be related to the increase in the non-Gaussian VaR over the period: as explained in Section 3.3.2, the decreasing trend of the yield over the sample implies an increase in the DUR-APP volatility and a decrease in expected return, that both contribute to increase the non-Gaussian VaR. In other words, the downside risk measure captures the increase in riskiness of bond investments as yield decrease.
3. Implementation of CRP Strategies

3.4 CRP III Strategies

The CRP III strategies are defined in Section 2.5 as MSR strategies. It is known that MSR weights are extremely sensitive to expected returns, so that robustification in the estimation of these parameters is all the more needed here. As shown in Proposition 4, the MSR portfolio of stock and bond achieves risk parity for volatility if both assets have the same Sharpe ratio. In order to take into account this condition in our estimation of expected returns, we deviate from the method described in Section 3.3.1 and impose that both Sharpe ratios used in the prior estimates are equal, as opposed to taking two different long-term values. Specifically, we construct the priors as:

$$\tilde{\mu}^\text{prior}_{St} = r_t + \bar{\lambda}_B \sigma_{B-DUR-\text{APP},t},$$

$$\tilde{\mu}^\text{prior}_{B} = r_t + \bar{\lambda}_B \sigma_{B-\text{RW},t},$$

that is, taking the same constant Sharpe ratio for the stock and the bond, and choosing this common value to be the long-term Sharpe ratio of the bond, equal to 0.26. The posterior estimates are then formed as in Section 3.3.1. The corresponding MSR strategy is referred to as MSR-SAME-SR. We also compute the weights for an MSR strategy in which the prior expected returns assume different long-term Sharpe ratios for the stock and the bond, that is respectively 0.41 and 0.26. The name of this strategy is abbreviated as MSR-DIFF-SR.

Figure 6 shows the weights of the MSR portfolios, which exhibit more variability, hence more turnover, than those of CRP II strategies (Figure 5) or CRP I portfolios (Figure 2). This higher level of turnover signal a higher sensitivity to time variation in parameters, in particular expected returns: this lack of robustness of MSR weights with respect to expected returns is a well-known property (Bloomfield et al., 1977). Table 3 reports the concordance rates between the bond weight in each strategy and the bond yield: the highest rate, 73.39%, is achieved by the MSR-DIFF-SR rule, but overall these rates are close to those displayed by CRP II strategies (Table 3). They are even slightly lower than that of the strategy based on non-Gaussian VaR.

To summarise, Table 5 shows the bond weight in the nine risk parity strategies considered in this section on the last date in the sample, which is 31 December 2012. This weight is taken as an indicator of each strategy's responsiveness to the long decrease in interest rates that took place between the early 1980s and 2012. All of these strategies assign a weight greater than 50% to the bond, but there are substantial differences between them. URP-VOL-RW and CRP-VOL-GARCH are by far the strategies that involve the largest bond allocation, around 80%; indeed, they rely on purely statistical measures of volatility, according to which the bond is less risky than the stock in spite of the decreasing trend followed by interest rates over the last 30 years. The two portfolios that rely on a duration-based volatility, namely CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP, have a significantly lower bond allocation, close to 71%; this decrease is due to the use of an alternative volatility measure, which is decreasing in the yield, either in a statistical (observed duration) or in a deterministic (approximate duration) way. The last five strategies depend on expected returns, either because they equalise the contributions to a downside risk measure (CRP-GSV, CRP-GVAR99 and
3. Implementation of CRP Strategies

CRP-NGVAR99), or because they favour by construction equities that have high expected returns (MSR-SAME-SR and MSR-DIFF-SR); those based on Gaussian risk measures (semi-volatility or VaR) do not substantially decrease the allocation to bonds with respect to CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP. It is the strategy that takes into account the lack of normality in returns, namely CRP-NGVAR99, that involves a lower allocation to bonds at the end of the period with a bond weight at only 53.33%.
4. Benefits of CRP Strategies in Periods of Rising Interest Rates
4. Benefits of CRP Strategies in Periods of Rising Interest Rates

The previous sections have shown that the various forms of CRP strategies tend to allocate less to the bond index in low yield environments compared to standard URP strategies, with the CRP-NGVAR99 strategy involving the lowest bond allocation of all tested strategies. The purpose of this section is to analyse the comparative performance of various risk parity strategies under the scenario of a mean-reversion of bond yields back up to their historical mean levels. We resort to a Monte-Carlo analysis in order to simulate an increase in interest rates starting from the historically low levels observed in December 2012. We test the URP-VOL-RW strategy and a fixed-mix as benchmarks, and the CRP strategies that we implement are CRP-VOL-DUR-APP and CRP-NGVAR99.19

4.1 Simulating an Increase in Bond Yield

The general principle in our Monte-Carlo analysis is to be consistent with the estimation methods for volatilities and expected returns as they have been outlined in Section 3. We thus have to simulate paths for the following processes:

- bond and stock prices;
- dividend-price (D/P) ratio;
- yield-to-redemption and duration;
- short-term rate.

In what follows, we describe the model and parameter assumptions that underlie our simulations, and we study the behaviour of the various risk measures and related risk parity strategies over the simulated period.

4.1.1 Description of the Monte-Carlo Model

The assumption that interest rates are mean reverting is critically important for the analysis conducted in this section. Hence, we model the yield-to-redemption of the bond index as an autoregressive process of order 1 (AR(1)), with an autoregression coefficient \( \alpha_\theta \) and an intercept \( \alpha_\theta^0 \).

Throughout Section 4, we use the same notation as in the previous ones for the different variables, but simulated values will be denoted with a tilde in order to distinguish them from the historical values (i.e. the values that relate to the sample period, January 1973-December 2012). The simulation step will be one month, and the simulation dates are 0, 1, 2, ... . We express all parameters in monthly terms. Letting \( \varepsilon_\theta \) be a Gaussian white noise, we have that the simulated yield evolves as:

\[
\tilde{\theta}_{t+1} = \alpha_\theta^0 + \alpha_\theta \tilde{\theta}_t + \sigma_\theta \varepsilon_{\theta,t+1}.
\]  

(4.1)

Bond duration is computed according to the approximation (2.10). The maturity \( m \) is taken equal to the value computed over the historical sample (see Section 3.2.1), that is \( m = 5.37 \) years.

To simulate bond returns consistent with the simulated yield and duration, a natural option is to use (3.5), which gives an approximation of the one-period return on a coupon bond selling close to par. Thus, the simulated monthly logarithmic return evolves as:

\[
\log(1 + \tilde{X}_{B,t,t+1}) = \tilde{D}^{\text{app}}_t (\tilde{\theta}_{t+1} - \tilde{\theta}_t) + \tilde{\delta}_t,
\]  

(4.2)

with \( \tilde{\delta} = \frac{1}{12} \).

The interest rate of maturity \( h \) (where \( h \) denotes the rebalancing period) is also

4. Benefits of CRP Strategies in Periods of Rising Interest Rates

simulated according to an autoregressive process:

$$
r_{t+1} = \alpha_r^0 + \alpha_r^1 r_t + \sigma_r \varepsilon_{r,t+1},
$$

where \( \varepsilon_r \) is a Gaussian white noise correlated with \( \varepsilon_\theta \).

Regarding the stock index, we take the expected excess return to be a piecewise constant function of time (it will be specified in Section 4.1.2). The simulated stock return thus evolves as:

$$
\log(1 + \tilde{X}_{S,t+1}) = \delta \tilde{r}_t + x_{St} + \sigma_{St} \varepsilon_{S,t+1},
$$

where \( x_{St} \) and \( \sigma_{St} \) are the (monthly) expected excess return and volatility for period \( [t, t+1] \).

The dividend-price ratio, denoted by \( DP \), is modelled as a mean-reverting process:

$$
\tilde{D}_{P,t+1} = \alpha_{DP}^0 + \alpha_{DP}^1 \tilde{D}_P + \sigma_{DP} \varepsilon_{DP,t+1},
$$

and the estimated expected return of the stock in the simulated dataset is constructed as in Section 3.3.1, that is, by combining an affine function of the D/P ratio with a prior. The coefficients of the affine function are taken equal to the historical estimates \( \alpha_S \) and \( \beta_S \).

4.1.2 Choice of Parameter Values

In order to simulate realistic paths, we base our choice of parameter values on the historical sample. For the D/P ratio, we estimate the coefficient \( \alpha_{DP}^0 \) as the slope in the regression of current value on previous month value: this estimate is close to unity (0.9915), which confirms that the D/P ratio is a highly persistent process.\(^{21}\)

The intercept \( \alpha_{DP}^0 \) is chosen in such a way that the long-term mean of the simulated D/P ratio is equal to the historical mean of the D/P ratio, which is 3.04%. Since the long-term mean of the process defined by (4.4) is

$$
\overline{DP} = \lim_{t \to \infty} E_0[DP_t] = \frac{\alpha_{DP}^0}{1 - \alpha_{DP}^1},
$$

we have \( \alpha_{DP}^0 = 0.00026 \). To fix the volatility \( \sigma_{DP} \), we match the long-term volatility of monthly changes in the D/P ratio implied by the dynamics (4.4) with the historical volatility of these changes, which is of 0.17%. It can be shown that this long-term volatility is

$$
\lim_{t \to \infty} \sqrt{\frac{\sigma_{DP}^2}{1 + \alpha_{DP}^1}} = \sigma_{DP},
$$

which implies \( \sigma_{DP} = 0.0017 \) (note that when the process is close to having a unit root, the long-term volatility is close to \( \sigma_{DP} \)).

All correlations between innovations are taken equal to the historical correlations between stock returns, yield changes, short rate changes and D/P ratio changes (see Table 1):

\[
\rho_{St} = -0.1158, \quad \rho_{DSt} = -0.8833, \\
\rho_{DPr} = 0.1889, \quad \rho_{Sr} = -0.0122, \\
\rho_{Df} = 0.7241, \quad \rho_{DP_r} = 0.0457.
\]

Moreover, all processes are initialised at their values on 31 December 2012, which is taken to be date 0 of the simulations:

\[
\hat{S}_0 = 3033.79, \quad \hat{D}_0 = 2064.56, \\
\hat{\theta}_0 = 0.0086, \\
\hat{\rho}_0 = 0.0005, \quad \hat{DP}_0 = 0.0227.
\]

To set the parameters of the yield-to-redemption and the stock process, we...
4. Benefits of CRP Strategies in Periods of Rising Interest Rates

Consider two economic scenarios. In the first scenario, interest rates grow progressively, without a significant impact on the equity market. Specifically, we assume that the long-term level of the yield is 6.63%, which is the historical mean, and that it takes on average five years for the yield to grow from its initial level to 6%, which represents about 90% of the long-term value. Letting \( b = 0.06 \), \( \theta = 0.0663 \) and \( t = 5 \times \frac{1}{\delta} \) (the factor \( \delta^{-1} \) arises because we simulate monthly values), we thus take the speed of mean reversion to be given by:

\[
\alpha_\theta^1 = \left( \frac{b - \theta}{\theta - \theta} \right)^{\frac{1}{t}},
\]

which implies \( \alpha_\theta^1 = 0.9638 \). On the equity market side, the annual risk premium and volatility stay equal to 6% and 15%, which amounts to taking \( x_{St} = 0.05 \) and \( \sigma_{St} = 0.043 \).

In the second scenario, interest rates increase at a much faster pace: it takes on average two years for the yield to grow from its current level to 6%. The corresponding speed of mean reversion is calculated through (4.5), taking \( t = 12 \times \frac{1}{\delta} \), which leads to a lower autoregression coefficient, namely \( \alpha_\theta^1 = 0.9120 \). In this scenario, the central bank fails at controlling inflation expectations, and there is a large negative impact on the equity market. Specifically, the monthly equity risk premium takes the following values:

\[
x_{St,2013} = 0.06 \times \delta, \quad x_{St,2014} = 0.00 \times \delta, \quad x_{St,2015} = -0.10 \times \delta, \quad x_{St,2016} = 0.05 \times \delta, \quad x_{St,2017} = 0.10 \times \delta.
\]

To reflect the empirical fact that stock volatility tends to increase in bear markets, we set it as follows:

\[
\begin{align*}
\sigma_{S,2013} & = 0.15 \times \sqrt{\delta}, \\
\sigma_{S,2014} & = 0.20 \times \sqrt{\delta}, \\
\sigma_{S,2015} & = 0.25 \times \sqrt{\delta}, \\
\sigma_{S,2016} & = 0.25 \times \sqrt{\delta}, \\
\sigma_{S,2017} & = 0.2 \times \sqrt{\delta}.
\end{align*}
\]

For years after 2017, we take \( x_{St} = 0.06 \times \delta \) and \( \sigma_{St} = 0.15 \times \sqrt{\delta} \).

In each scenario, the intercept \( \alpha_\theta^0 \) in (4.1) is taken equal to \( \theta \times (1 - \alpha_\theta^1) \), so that the long-term mean of the simulated yield coincides with the historical mean. We also set the volatility equal to \( \sigma_{St} \times \sqrt{\frac{1 + \alpha_\theta^0}{2}} \), where \( \sigma_{\theta} = 0.0043 \) is the historical volatility of yield changes. This ensures that the long-run volatility of simulated yield changes coincides with the historical value. Regarding the short-term interest rate, the speed of mean reversion is set to the same value as that of the index yield in each scenario, and the intercept \( \alpha_\theta^0 \) and the volatility \( \sigma_r \) are adjusted in order to match the historical values, as for the other two mean-reverting processes.

This calibration process leads to two sets of parameter values, that correspond to two economic scenarios for interest rates and the equity market. In what follows, we refer to the scenario with slow increase in interest rates as “scenario 1”, and the one with rapid increase and strong impact on the equity market as “scenario 2”. In each scenario, we simulate 2,000 paths for the stock and bond prices, the D/P ratio, the yield and the short-term rate over fifteen years, that is, from 31 December 2012 to 31 December 2027.

4.1.3 Properties of Simulated Paths

We first look at the simulated paths from a descriptive standpoint. Figure 7 shows the path followed by the yield between
January 1973 and December 2012, and the average of simulated yields across the 2,000 paths, between December 2012 and December 2027, for both scenarios. Since the yield starts from 0.86% on 31 December 2012, and the speed of mean reversion is of 6.63%, the simulated yield follows an increasing trend in the simulated dataset. At the initial date, the yield is 0.86%. After five years, in December 2017, the average yield is 6.04% in scenario 1, which is in line with our parameter choices. This value is 0.59% below the long-term value, so mean reversion is almost complete. In scenario 2, mean reversion is much faster: the average yield in December 2014 is 5.98%, which is already very close to 6%, and it is virtually indistinguishable from 6.63% as of December 2017. Note that the long-term value is an absorbing state, so once the average yield has reached it, it stays constant at this level. It should also be emphasised that Figure 7 shows an average behaviour across simulations, and that the yield does not necessarily follow this pattern on a given path. In particular, the actual time needed to revert back to mean can vary from one path to the other. The figure also shows the historical and the average simulated durations. As duration is computed as a decreasing function of the yield, it exhibits a pattern which is opposite to that of the yield. In particular, it follows an increasing trend over the historical period, and a decreasing trend over the simulated one.

Table 6 allows for a comparison between the historical path and the simulated paths by reporting descriptive statistics on the two datasets: this allows us to check that the simulator produces "realistic" paths. For the sake of brevity, we only report the statistics that relate to scenario 1, but qualitatively similar results would be obtained for scenario 2. By definition, the numbers in upright font are the same as those in Table 1, and those in italic are averages of statistics across the 2,000 paths. Historical volatilities of yield and D/P changes are perfectly replicated by the model (see Panel (b)), because the volatilities of these processes have been set in such a way as to achieve this match. Stock volatility is not as accurately replicated, although \( \sigma_3 \) was fixed at the historical volatility. This is because part of the variance of simulated stock returns comes from the randomness of the total expected return (see (4.3)); although the risk premium is constant across states of the world, the short-term rate is not, hence the drift also contributes to the realised volatility. Unlike the volatilities of differences, the volatilities of the yield and the D/P ratio in level are not well reproduced, which suggests that an AR(1) model lacks the sufficient flexibility to describe these processes, even though there is no reason to believe that this induces any particular bias in the the analysis. The other second-order moments, namely the correlations, are overall well replicated (Panel (f)). Some signs are flipped with respect to the historical values, but they correspond to correlations that are close to zero. Hence, the simulator neither "misses" correlations that were present in the original data, nor it "creates" spurious correlations between variables. Nevertheless, it tends by construction to reinforce the correlations between bond returns, yield changes and duration changes: these three variables have fairly strong correlations in the sample, but these effects are more pronounced in the simulation because duration is modelled as a function of yield, and because simulated
bond returns are explicitly functions of yield changes (see (4.2)).

Looking at mean values in Panel (a), we note that although they are not at odds with respect to historical values, the simulator does not replicate them as well as second-order moments. This is because sample means are much more sensitive to the choice of the sample than volatilities and correlations. In particular, the fact that the yield progressively increases in the simulated dataset implies that the bond index performs less well than in the historical sample. The simulated Sharpe ratio of the bond is even negative, because the bond underperforms the cash account. Finally, the only properties of the original data that the model fails to capture is the presence of skewness and kurtosis. Unsurprisingly, the assumption of Gaussian innovations produces skewness and kurtosis in bond and stock returns that are close to zero, which is clearly at odds with the data.

4.2 Risk Measures for Bond and Stock Indices
Having simulated the returns, we can compute the volatilities and the non-Gaussian VaR, computed with the DUR-APP volatility for the bond and the RW volatility for the stock. Skewness and kurtosis are set to the same values as in the historical dataset (see Section 3.3.2). The left part of each curve (i.e. before December 2012) is identical to Figure 1, and Figure 4 for downside risk measures. As mentioned above, RW volatility is lower than DUR-APP volatility, except in the beginning of the sample, where it was the higher of the two measures. On the other hand, DUR-APP volatility follows an increasing trend, which is due to the decrease in bond yield. This pattern is also observed in the non-Gaussian VaR, where it is accentuated by the fact that this measure depends on the expected returns, which tends to decrease over the period. Hence, in December 2012, the VaR measure recognises that downside risk is substantial, an element which is missed by the RW volatility measure.

In the simulated sample, the DUR-APP volatility decreases, thereby following the same path as the duration (see Figure 7). The bond index VaR is also negatively related to the yield, through the volatility and the expected return, which implies that it tends to decrease after December 2012, as the yield goes up. Remarkably, the bond index RW volatility equals the DUR-APP volatility on average, after a period of approximately five years (even though the two measures are not in perfect agreement for each path). This property arises because by construction (see (4.2)), the true bond index volatility in the simulations is close to the DUR-APP volatility.

The stock index RW volatility converges to a constant value in the simulation, which is
the 15.78% per year implied by the model (4.3). The downside risk measures also stabilise after a few years. The limit values are close to the values of the risk measures computed with the long-term expected return and volatility. With \( \rho = \frac{\sigma_S}{1 - \alpha_S^2} \), the long-term expected return is:
\[
\frac{1}{12} (\rho + 0.15) = 0.0094,
\]
and the long-term volatility is \( \sigma_S = 0.0433 \), which implies a VaR of 11.76, close to the value at which the average VaR stabilises.

### 4.3 Performance of Strategies

We now turn to the comparative analysis of the CRP strategies, as well as two benchmark strategies which are the URP-VOL-RW strategy and a fixed-mix policy. All of these are invested in the stock index and the bond index only. As a reminder, let us note that URP-VOL-RW is a volatility-RP strategy, which uses five-year rolling-window estimates for the two volatilities. CRP-VOL-DUR-APP is a volatility-RP strategy that also takes stock volatility equal to the RW volatility, but it differs from URP-VOL-RW in that it uses a volatility proportional to the duration for the bond index (see (2.9)). CRP-NGVAR99 equalises the contributions of the two assets to the non-Gaussian VaR at 99%. Finally, there are two MSR strategies, which use different priors for stock and bond expected returns: in MSR-DIFF-SR, the priors respectively imply a Sharpe ratio of 0.26 for the bond and 0.41 for the stock (these are the long-term historical Sharpe ratios of US stocks and Government bonds), while in MSR-SAME-SR, both priors correspond to a Sharpe ratio of 0.26. The fixed-mix strategy, referred to as AVG-URP-VOL-RW, has weights equal to the average weights of URP-VOL-RW over the historical period (see Figure 2), that is 74.04% for the bond, and 25.96% for the stock. This bond weight comprised between that of URP-VOL-RW and those of CRP strategies (see Table 5).

Table 7 reports simulated performance statistics year by year between 2013 and 2017, for the two economic scenarios. In Panels (a) and (c), the first line contains the average yield for each year, and the other lines contain the average excess returns (over the risk-free rate) for the two constituents (stock index and bond index), and for the various RP strategies under consideration. All statistics are computed from monthly observations, and are averaged across the 2,000 paths. The annual average yield increases as time goes by, and it does so at a decreasing marginal rate, especially in scenario 2, where the rise is sharp in the first two years; this decreasing speed corresponds to the flattening of the curve in Figure 7. In parallel with this interest rate growth, the bond experiences negative returns. In scenario 1, where the yield goes up at a relatively slow rate, excess returns are strongly negative in the first two years (e.g. \(-9.7\%\) in 2013), and they then stay negative until 2017, although they are close to zero this year. In scenario 2, losses are much more severe in the first years, with the bond index underperforming the cash account by \(18.4\%\) in 2013. On the other hand, the bond index recovers more quickly. It may seem puzzling that the yield keeps increasing in 2016 and 2017, while the bond has positive excess returns of 0.7% and 1% in these two years, but this is because the yield increase takes place at a very slow rate, while the bond keeps paying coupons. Hence, the income term in (3.5) is positive, which...
compensates the negative contribution of the yield change term. In Panels (b) and (d), we report the annual volatilities of the yield, the two assets and the various strategies: bond volatility slightly decreases as the yield approaches its long term mean. As far as the stock is concerned, the annual risk premia and volatilities shown in the table are by construction close to the values that were set in the simulations.

In both scenarios, all strategies have negative returns in 2013, due to the adverse performance of bonds. Unsurprisingly, it is the two strategies that were most exposed to bonds in December 2012, namely URP-VOL-RW and AVG-URP-VOL-RW, that display the most negative returns. The dispersion among the expected returns subsequently shrinks as the yield converges to its long-term mean value. For instance in scenario 1, the difference between the highest and the lowest expected return is 3.9% in 2013, versus 1.1% in 2017. In scenario 1, the top performer every year from 2013 to 2016 is CRP-NGVAR99; this is because this strategy has the lowest exposure to bonds, and it benefits from the stable performance of the equity index. However, in scenario 2, it has the worst performance in 2015, underperforming the benchmark strategies by more than 1%. Indeed, that year is characterised by a strong equity bear market, so that the higher stock weight penalises the portfolio performance. Similarly, CRP-VOL-DUR-APP has the second lowest return in 2015, because the corresponding portfolio contains more stocks than the other strategies, except CRP-NGVAR99. 2015 also appears as a bad year for CRP-VOL-DUR-APP and CRP-NGVAR99 from a risk standpoint, because these strategies are impacted by the large volatility of stocks. But even in the two years with negative or low equity risk premium (2014 and 2016), CRP-NGVAR99 outperforms the other strategies.

Overall, these numbers illustrate in a quantitative manner the fact that overweighting bonds after a long period of decrease in interest rates, as a typical risk parity approach based on historical volatility does, leads to strongly negative returns when interest rates start to increase back to higher levels, especially if this increase is rapid. On the other hand, due to their larger exposure to the equity market, CRP strategies are of course also sensitive to the impact of the interest rate rise on the equity risk premium and volatility. In particular, they display stable outperformance with respect to the benchmark policies only if the increase in rates takes place at a sufficiently low speed so as not to have a strong negative impact on the equity market.

4. Benefits of CRP Strategies in Periods of Rising Interest Rates
5. Introducing Commodities
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In this section, we introduce a broad commodity index as a third asset class. We take it to be the Goldman Sachs Commodity Index (GSCI), which is an investable index representing the return of an investment in commodity futures contracts. We repeat the analysis based on historical data conducted in Section 3, checking the robustness of the benefits of CRP strategies with respect to a URP strategy based on historical volatilities when a third asset class is present.

5.1 Estimating Commodity Volatility and Expected Return

We first estimate commodity volatility using a five-year rolling window and a GARCH(1,1) model. We follow the procedure described in Section 3.2.1. Figure 1 shows the two measures of volatility, expressed in annual terms. As was the case for the stock volatility, the GARCH measure shows a significantly higher variability than the RW measure. However, the average values for the two measures are very close to each other, being 18.7% for the RW-measure and 19.0% for the GARCH measure. Both time series of volatility show two large spikes, one corresponding to the October 1990 oil price spike, the second to the November 2008 huge oil price drop. These sudden increases in volatility measure are quickly smoothed out in the GARCH volatility, while they persist for a five-year period with the RW volatility measure. But unlike volatility-based RP portfolios invested in stocks and bonds only, CRP type I strategies that include commodities do not depend solely on volatilities, and they also require the estimation of the three pairwise correlations between stocks, bonds and commodities. Again, similarly to the methodology previously followed, we estimate the pairwise correlations between assets by computing the sample correlations of log-returns over a five-year rolling window. As can be noticed in Figure 10, the correlation between the stock index and the bond index exhibits two main regimes, a regime of positive correlation going from the beginning of the sample to the late 1990s, followed by a long period of negative correlation. For what concerns the correlation between the commodity index and the stock and bond indices, there are multiple variations of sign and it is more difficult to identify different correlation regimes.

For the implementation of CRP type II strategies, it is also necessary to provide estimates for the expected return on the commodity index. Several papers have analysed the possibility of predicting movements in commodity spot prices and futures excess returns. Some of them analyse return predictability by considering the returns of one specific commodity at a time. Fama and French (1987) test the predictive power of the basis of futures contracts on individual commodities with respect to price changes and risk premia. The predictive power depends on the commodity under consideration, but it is on average rather low, especially for what concerns risk premia predictability. On the other hand, their analysis does not include the energy sector, which is the sector with the highest weight in the GSCI basket (almost 70% at the beginning of 2014). Melolinna (2011) studies the time-series predictability of the excess returns of WTI crude oil futures contracts by testing three different risk premia models, which consider as predictors the US stock
level, the net long position of speculators, the rolling-window correlation with the stock market and the interest rate. Their conclusion is that these models are able to predict futures excess returns, but only over specific periods of time, as none of them is able to provide consistently acceptable results over the whole time period considered (1989 - 2008).

There is also a vast literature about commodity cross-sectional return predictability, aiming at building portfolios of commodity futures with risk-adjusted outperformance potential. For example, Gorton et al. (2013) build different portfolios of commodity futures, respectively based on inventory levels, futures basis and momentum of individual commodities, finding that all of these observables carry information allowing to generate excess returns. They also perform predictive regressions of excess returns of commodity futures over an indicator of inventory level, showing rather poor predictability at the individual commodity level. Basu and Miffre (2013) instead support the idea that the hedging pressure, that is the position taken by large hedgers, is a good predictor for future excess returns. The rationale is that large hedgers are willing to pay a premium to be able to hedge their risky positions with futures. These hedging positions are reported to the Commodity Futures Trading Commission and are made publicly available, which allows them to build an indicator as the ratio between the net short hedge positions and the total number of both short and long hedge positions. Based on this indicator, they build long-short portfolios that outperform the typical commodity benchmarks. Overall, while these papers suggest methods to capture excess return by wisely constructing portfolios from a basket of commodity futures, they do not support the idea that predicting risk premia at the individual commodity level or at the index level is effectively possible. Finally, Gargano and Timmermann (2012) focus on commodity spot price in- and out-of-sample predictability over monthly, quarterly and annual horizons. They test all the predictors used for stock return predictability by Welch and Goyal (2008), as well as lagged commodity returns, three macroeconomic indicators (industrial production growth, change in unemployment rate, growth in monthly money stock), and a commodity volatility measure. They show that, especially for short predictive horizons, commodity returns are strongly persistent. Macroeconomic variables have a strong predictive power over annual spot price changes. However, they do not test the predictive power of these observables with respect to futures risk premia.22

Another approach to model commodity spot and futures prices dynamics is proposed by Casassus and Collin-Dufresne (2005), who build an arbitrage-free dynamic model with three factors to describe the term structure of interest rates, commodity spot prices and the convenience yield, allowing for time-varying risk premia. Convenience yields and risk premia are functions of the three state variables, which captures the dependence of these variables with respect to the level of bond yields and the commodity spot price. Model parameters are estimated maximum likelihood for four individual commodities (oil, copper, gold and silver), using data on the US yield curve and futures commodity prices. We retain from their empirical application the
use of the interest rate level and the first two principal components of convenience yields as predictors for the return of the commodity index. The variable that we seek to predict is the total return on the GSCI index, but futures contracts written on the GSCI spot index are illiquid and available only over a very short period of time. For this reason, as the energy sector is dominant in the GSCI and WTI crude oil is the dominant commodity in the energy sector, we decide to use WTI crude oil futures data to extract the convenience yields for different maturities. Unfortunately, our analysis shows a very poor predictive power of the combination of the three potential predictors with respect for the GSCI.

Eventually, given the inconclusive results from the literature, which are confirmed by our own analysis of commodity return predictability, we make the assumption that the commodity index has a constant Sharpe ratio, which we take equal to the value measured from monthly data from December 1969 to December 2012, namely $\lambda_C = 0.22$. The expected return is thus estimated as:

$$\hat{\mu}_C = \hat{\mu}_C^{\text{prior}} = r_t + \lambda_C \hat{\sigma}_C^{\text{RW}} W_t.$$  

This procedure is equivalent to that followed in Section 3.3.1 for constructing stock and bond expected returns, but with a full shrinkage towards the non-informative prior.

### 5.2 Risk Parity Strategies

When the asset universe is constituted by more than two assets, even the weights for the CRP I strategy (that is, the one based on volatility) cannot be computed in closed form. Thus, we numerically solve the system (2.1) (for the volatility risk measures) or the system (2.3) (for the dissymmetric risk measures).²³

Figure 11 shows the portfolio weights for the volatility-RP strategies. As for the two-asset case, we notice that the risk parity portfolios are dominated most of the time by the bond index, especially for what concerns URP-VOL-RW and CRP-VOL-GARCH. However, the introduction of commodities reduces the bond allocation with respect to Figure 2, as it allows the total risk of the portfolio to be diversified over more assets. The CRP-VOL-GARCH strategy has again the highest turnover, which is a direct consequence of the large variability of the GARCH volatility measures. The conditional risk parity strategies relying on duration-based volatility measures, namely CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP, have a lower bond allocation when the bond yield is low. In Table 8, which shows bond allocation on 31 December 2012, it can be seen that the weight in the bond index drops from 77.6% for URP-VOL-RW to 68.0% and 68.4%, respectively, for CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP.

Turning to the relationship between portfolio weights and bond yield levels, Table 9 shows that, for the RW and GARCH volatility measures, the correlations are negative and close to $-60\%$, while they are closer to zero for the volatility measures based on bond duration. These values are lower than those obtained in the two-asset universe. The concordance rates between monthly changes in portfolio weights and yield are instead in line with those obtained for two assets only, thus being significantly higher for the strategies CRP-VOL-DUR-OBS.

---

²³ - We have checked that the solution to (2.1) agrees with the solution derived by minimising volatility subject to an inequality constraint over the sum of the logarithms of the portfolio weights, which is one of the algorithms introduced by Maillard et al. (2010) to compute volatility-RP portfolios.
5. Introducing Commodities

and CRP-VOL-DUR-APP. Even though the introduction of a third asset introduces more variability in the bond weight, these strategies still tend to increase the bond allocation when yields are high and bonds are inexpensive, while reducing it when yields are lower and bonds more expensive.

The weights of the CRP II strategies are shown in Figure 12. As for the two-asset case, the portfolio weights of the strategies relying on Gaussian risk measures (CRP-GSV and CRP-GVAR99) look very much similar to those obtained for the CRP-VOL-DUR-APP portfolio (see Panel (d) in Figure 11), even though they have higher correlations between bond portfolio weights and bond yield, as well as a higher concordance rate between changes in weight and yield, as can be seen in Table 9. This is again because the use of a dissymmetric risk measure introduces a further dependence, through the bond expected return, of the risk measure to the current level of the yield. Relying on a non-Gaussian risk measure, in this analysis the CRP-NGVAR99 strategy also has a significantly different pattern of portfolio weights, with a reduced allocation to the bond index when the yield is low. This strategy also has a much higher correlation between bond portfolio weight and yield, which increases from below 10% to 67% and the highest concordance rate between weight and yield changes.

Finally, Figure 13 shows the weights of CRP III strategies. As for the two-asset case, the strategy MSR-SAME-SR assumes the same long-term Sharpe ratio for the prior expected returns of the three assets, namely the long-term Sharpe ratio of Government bonds (0.26). The strategy MSR-DIFF-SR considers instead three different long-term Sharpe ratios, which are $\lambda_B = 0.26$, $\lambda_S = 0.41$ and $\lambda_C = 0.22$. Although the turnover is visibly lower than in the two-asset case, it is still higher than for CRP I and CRP II strategies. It is interesting to see that in the early 1980s, the stock weight drops to zero in favour of the commodity index, probably because commodities have a low volatility in this period (see Figure 9), as well as a low correlation with bonds. The high correlation between bonds and stocks recorded in those years also contributes to tilt stocks out of the portfolio (see Figure 10). The bond allocation on 31 December 2012 (Table 8) is in line with those of CRP II strategies based on Gaussian risk measures. In terms of concordance between bond weight and yield (Table 9), MSR strategies with commodities display slightly better results than CRP strategies based on Gaussian measures, which is in line with the results obtained in the two-asset case. However, the best correlation and concordance rate between the weight and the predictor are still produced by the CRP-NGVAR99 strategy.
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The traditional approach to constructing risk parity portfolios uses rolling-window estimates for volatilities, which raises a number of concerns that our paper attempts to analyse. The first concern is of statistical nature: these volatility estimates depend on a particular series of observed returns, which exposes the investor to sample risk. The second concern is that the true values for the risk parameters are not stationary: rolling-window estimates reflect past true volatility levels, but do not necessarily correspond to the conditional volatility of future returns. While this second concern can in principle be alleviated by replacing rolling-window volatility by a GARCH volatility estimate, our results suggest that the two risk parity strategies have overall similar properties. In this paper, we introduce an alternative bond volatility estimate, which relies on the model-free approximation of a bond return as the product of (the negative of) duration times the change in yield-to-redemption. This volatility measure offers the advantage of being instantaneously observable. The third and last concern raised by the use of historical volatility in standard risk parity strategies is related to the choice of volatility as a risk measure. Indeed, volatility does not disentangle downside risk from upside risk. This is a particularly serious concern in the current low yield environment, with interest rates that have been decreasing since the early 1980s, to reach historically low levels after the 2008 financial crisis. While such low levels of interest rates signal an increase in downside risk, historical volatility of bonds has not increased in parallel.

In this context, we analyse "conditional risk parity" strategies, that is strategies that are more responsive to changes in market conditions in general, and yield levels in particular. In response to the first two major problems identified with historical volatility, namely sample dependency and backward-looking bias, we introduce in this paper an alternative bond volatility measure, which we refer to as “duration-based volatility” (in short, DUR volatility). This measure is suggested by the model-free approximation of the return on a bond portfolio as the product of the negative of duration times the yield change. Our empirical analysis confirms that the DUR volatility measure has followed an increasing trend due to the decreasing trend in bond yield levels. To reinforce this effect and address the third concern with standard risk parity strategies, one may replace volatility by a downside risk measure, such as semi-volatility or Value-at-Risk. In this context, we propose to use the Cornish–Fisher VaR, which incorporates information on stock and bond return skewness and kurtosis. Among the conditional risk parity strategies that we test, we find that the one that equalises the contributions to this non-Gaussian VaR is the strategy that implies the lowest bond allocation in a low yield environment, and also that it is the one for which the bond allocation is the most likely to decrease after a decrease in interest rates.

In spite of these advantages, conditional risk parity strategies have their own challenges. In particular, these strategies are more demanding than unconditional risk parity in terms of parameter estimation. The strategy based on non-Gaussian VaR requires co-skewness and co-kurtosis parameter inputs, which are well known to dramatically increase the number of parameters as the investment universe
6. Conclusion

grows. This concern, however, should be of limited importance if these strategies are employed in an asset allocation context, with a limited number of asset classes. More problematic is the estimation of expected returns, in view of the notorious lack of robustness of statistical estimates of these quantities. In this paper, we have proposed a relatively rough way to address this issue, by shrinking forecasted returns towards a prior, but the literature has provided numerous methods to improve expected return estimates. It would be interesting, and practically relevant, to assess the sensitivity of conditional risk parity portfolios to estimation errors in expected returns, and to study the benefits of explicitly taking into account parameter uncertainty, even if our analysis suggests that the impact of errors in expected return estimates are much less pronounced compared to what the case is with standard mean-variance portfolio optimisation.
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Appendices
A. Proofs of Propositions

A.1 Proof of Proposition 2

We let \( \tau_i = t_i - t \). To alleviate the notations, we write \( \sum_{i, t_i > t} \). We have:

\[
D_t = \frac{\partial D_t}{\partial \theta_t} = P_t \left[ \sum_i h_i \tau_i^2 e^{-\tau_i \theta_t} - \tau_m^2 F e^{-\tau_m \theta_t} \right] \left[ \sum_i h_i e^{-\tau_i \theta_t} + F e^{-\tau_m \theta_t} \right]
+ \left[ \sum_i h_i \tau_i e^{-\tau_i \theta_t} + \tau_m F e^{-\tau_m \theta_t} \right]^2.
\]

Expanding the products, we get:

\[
P_t \frac{\partial D_t}{\partial \theta_t} = - \sum_{i,j} h_i h_j \tau_i^2 e^{-(\tau_i + \tau_j) \theta_t} + \sum_{i,j} h_i h_j \tau_i \tau_j e^{-(\tau_i + \tau_j) \theta_t}
- \tau_m^2 F e^{-\tau_m \theta_t} \sum_i h_i e^{-\tau_i \theta_t} - F e^{-\tau_m \theta_t} \sum_i h_i \tau_i^2 e^{-\tau_i \theta_t}
+ 2 \tau_m F e^{-\tau_m \theta_t} \sum_i h_i \tau_i e^{-\tau_i \theta_t}.
\]

The sum of the two terms in the first line of the right-hand side can be rewritten as:

\[
\sum_{i,j} h_i h_j e^{-(\tau_i + \tau_j) \theta_t} [\tau_i \tau_j - \tau_i^2] = \sum_{i<j} h_i h_j e^{-(\tau_i + \tau_j) \theta_t} [\tau_i \tau_j - \tau_i^2]
+ \sum_{i<j} h_i h_j e^{-(\tau_i + \tau_j) \theta_t} [\tau_i \tau_j - \tau_j^2]
= - \sum_{i<j} h_i h_j e^{-(\tau_i + \tau_j) \theta_t} [\tau_i - \tau_j]^2.
\]

The sum of the three terms in the second line of the right-hand side of (A.1) can be rewritten as:

\[-F e^{-(t_{m-1}) \theta_t} \sum_i e^{-\tau_i \theta_t} [\tau_i^2 + \tau_i^2 - 2 \tau_i \tau_i] = -F e^{-(t_{m-1}) \theta_t} \sum_i e^{-\tau_i \theta_t} [\tau_i - \tau_i]^2.
\]

Hence:

\[
P_t \frac{\partial D_t}{\partial \theta_t} = - \sum_{i<j} h_i h_j e^{-(\tau_i + \tau_j) \theta_t} [\tau_i - \tau_j]^2 - F e^{-(t_{m-1}) \theta_t} \sum_i e^{-\tau_i \theta_t} [\tau_i - \tau_i]^2.
\]

Both terms in the right-hand side are negative when the bond has two or more cash-flows to pay (that is, when \( t < t_{m-1} \)), so the partial derivative is negative too. When \( t_{m-1} < t < t_m \), the expression for the duration simplifies to:

\[
D_t = \frac{1}{F e^{-(t_{m-1}) \theta_t} \times (t_m - t) F e^{-(t_{m-1}) \theta_t}} = t_m - t.
\]
A.2 Proof of Proposition 3

We first compute the partial Laplace transform of a random variable Z distributed as \( \mathcal{N}(\mu, \sigma^2) \). Let s be a real number. We have:

\[
\mathbb{E}[e^{-sZ}1_{\{Z \leq a\}}] = \int_{-\infty}^{a} e^{-sx} n(x) \, dx.
\]

After a few algebraic manipulations, we obtain:

\[
\mathbb{E}[e^{-sZ}1_{\{Z \leq a\}}] = e^{-\mu s + \frac{s^2 \sigma^2}{2}} \mathcal{N}\left( \frac{\mu - \mu}{\sigma} + \sigma s \right).
\]

Differentiating both sides with respect to s, we get:

\[
\mathbb{E} \left[ Z 1_{\{Z \leq a\}} \right] = \mu \mathcal{N}\left( \frac{\mu - \mu}{\sigma} \right) - \sigma n \left( \frac{\mu - \mu}{\sigma} \right);
\]

\[
\mathbb{E} \left[ Z^2 1_{\{Z \leq a\}} \right] = (\sigma^2 + \mu^2) \mathcal{N}\left( \frac{\mu - \mu}{\sigma} \right) - [2\mu \sigma + (\mu - \mu) \sigma] n \left( \frac{\mu - \mu}{\sigma} \right),
\]

hence:

\[
\mathbb{E} \left[ (Z - a)^2 1_{\{Z \leq a\}} \right] = [\sigma^2 + (\mu - a)^2] \mathcal{N}\left( \frac{\mu - \mu}{\sigma} \right) - [(\mu - \mu) \sigma + 2\sigma(\mu - a)] n \left( \frac{\mu - \mu}{\sigma} \right).
\]

The Gaussian semivariance with \( a = 0 \) can be rewritten as:

\[
(\text{GSV}_0^0)^2 = \sigma^2 \left[ (1 + \ell^2) \mathcal{N}(\ell) - \ell n(\ell) \right], \tag{A.2}
\]

where \( \ell = \mu/\sigma \). Denoting with \( f(\ell) \) the expression enclosed within the brackets, we have:

\[
f'(x) = 2[x, \mathcal{N}(\ell) - n(\ell)], \quad f''(x) = 2\mathcal{N}(\ell).
\]

Hence, \( f''(x) < 0 \) and \( f'(\infty) = 0 \), so that \( f' \) is negative. This shows that \( (\text{GSV}_0^0)^2 \) is strictly decreasing in \( \ell \), hence in \( \mu \). Because \( (\text{GSV}_0^0)^2 = \sqrt{(\text{GSV}_1^0)^2} \), the Gaussian semivolatility is strictly decreasing in \( \mu \) too.

By (A.2), we have:

\[
\frac{\partial (\text{GSV}_0^0)^2}{\partial \sigma} = 2 \sigma f(\ell) + \sigma^2 \left( -\frac{\mu}{\sigma^2} \right) f'(\ell) = 2\sigma f(\ell) - \mu f'(\ell).
\]

As shown above, \( f'(\ell) \) is negative. Because \( f(\ell) \) is positive, it follows that \( \frac{\partial (\text{GSV}_0^0)^2}{\partial \sigma} \) is positive if \( \mu \) is positive.
A.3 Proof of Proposition 4

Consider the optimisation program (2.16). The first-order optimality conditions read:

\[
\frac{1}{\sigma_P t} \tilde{\mu}_t - \frac{\mu_P t}{\sigma_P^2 t} \Sigma_t w_t - \nu 1 - \eta = 0,
\]

\[
\eta \leq 0, \quad w_t \odot \eta = 0.
\] (A.3)

From the budget constraint \( w_t^t 1 = 1 \), it follows that \( \nu = 0 \), so that:

\[
\Sigma_t w_t = \frac{\sigma^2_P t}{\mu_P t} \tilde{\mu}_t - \frac{\sigma^3_P t}{\mu^2_P t} \eta,
\]

and the contributions to volatility are:

\[
c^\text{vol}_t(w_t) = \frac{\sigma_P t}{\mu_P t} w_t \odot \tilde{\mu}_t.
\]

If the MSR achieves risk parity (for volatility), then it must be that all weights are positive (if there were a zero weight, all contributions would be zero, and the portfolio would have zero volatility), so that \( \eta = 0 \). Hence \( w_t = \frac{\sigma_P t}{\mu_P t} \Sigma_t^{-1} \tilde{\mu}_t \), so that:

\[
c^\text{vol}_t(w_t) = \frac{\sigma^3_P t}{\mu_P t} \tilde{\mu}_t \odot \Sigma_t^{-1} \tilde{\mu}_t,
\]

which implies that \( \tilde{\mu}_t \odot \Sigma_t^{-1} \tilde{\mu}_t = x 1 \), for \( x = \frac{\sigma^3_P t}{\mu^2_P t} \).

Conversely, let us assume that \( \Sigma_t^{-1} \tilde{\mu}_t \geq 0 \) and that \( \tilde{\mu}_t \odot \Sigma_t^{-1} \tilde{\mu}_t = x 1 \) for some positive \( x \). Then \( \Sigma_t^{-1} \tilde{\mu}_t \) is actually positive, so that the portfolio \( w_t = \Sigma_t^{-1} \tilde{\mu}_t \) is well defined and is the MSR portfolio, since it satisfies the first-order conditions (A.3) for \( \nu = 0 \) and \( \eta = 0 \). Moreover, the contributions to volatility read:

\[
c^\text{vol}_t(w_t) = \frac{1}{|1' \Sigma_t^{-1} \tilde{\mu}_t| \sqrt{\tilde{\mu}_t \Sigma_t^{-1} \tilde{\mu}_t}} \tilde{\mu}_t \odot \Sigma_t^{-1} \tilde{\mu}_t = \frac{x}{|1' \Sigma_t^{-1} \tilde{\mu}_t| \sqrt{\tilde{\mu}_t \Sigma_t^{-1} \tilde{\mu}_t}} 1,
\]

so that they are all equal. We note that the weights of the MSR are inversely proportional to expected returns.

Thus, the MSR portfolio equals the RP one if, and only if:

\[
\begin{cases}
\Sigma_t^{-1} \tilde{\mu}_t \geq 0, \\
\tilde{\mu}_t \odot \Sigma_t^{-1} \tilde{\mu}_t = x 1
\end{cases}
\] (A.4)

Introducing the diagonal matrix of volatilities, \( \Sigma_t \) (size \( N \times N \)), the correlation matrix \( \Omega_t \) (size \( N \times N \)), and the vector of Sharpe ratios, \( \Lambda_t \) (size \( N \times 1 \)), we have:

\[
\tilde{\mu}_t = V_t \Lambda_t, \quad \Sigma_t = V_t \Omega_t V_t,
\]
so that:

\[ \Sigma_t^{-1} \mu_t = V_t^{-1} \Omega_t^{-1} \Lambda_t, \quad \bar{\mu}_t \odot \Sigma_t^{-1} \mu_t = \Lambda_t \odot \Omega_t^{-1} \Lambda_t. \]

Hence, (A.4) is equivalent to:

\[
\begin{cases}
\Omega_t^{-1} \Lambda_t \geq 0, \\
\Lambda_t \odot \Omega_t^{-1} \Lambda_t = x \mathbf{1} \text{ for some positive } x
\end{cases}
\tag{A.5}
\]

We now look for sufficient conditions for (A.5) to be satisfied. Assume that all assets have the same pairwise correlation, \( \rho_t \), and the same Sharpe ratio, \( \lambda_t \). Letting \( I \) be the \( N \times N \) identity matrix and \( J \) be the \( N \times N \) matrix of ones, we have:

\[ \Omega_t = J + \rho_t [J - I], \quad \Lambda_t = \lambda_t \mathbf{1}. \]

(Note that we must have \( 1 + (N - 1) \rho_t > 0 \) for \( \Omega_t \) to be positive definite.) Hence \( \Omega_t \mathbf{1} = 1 \Omega_t \mathbf{1} = \frac{1}{1 + (N - 1) \rho_t} \mathbf{1} \), and \( \Omega_t^{-1} \mathbf{1} = [1 + (N - 1) \rho_t] \mathbf{1} > \mathbf{0} \). Thus:

\[ \Omega_t^{-1} \Lambda_t = [1 + (N - 1) \rho_t] \lambda_t \mathbf{1}, \quad \Lambda_t \odot \Omega_t^{-1} \Lambda_t = \lambda_t^2 [1 + (N - 1) \rho_t] \mathbf{1}, \]

so that conditions (A.5) are satisfied.

A.4 Proof of Proposition 5

The contributions to the risk measure \( R_t \) defined in Proposition 5 are given by:

\[ c_t^R(w_t) = w_t \odot \left( -\bar{\mu}_t + \frac{\lambda_t \mathbf{MSR}}{\sigma_t} \sum_i w_i \right). \]

As shown in Appendix A.3, the MSR satisfies the following first-order optimality conditions:

\[
\frac{1}{\sigma_t} \bar{\mu}_t - \frac{\bar{\mu}_t \mathbf{MSR}}{\sigma_t^2} \sum_i w_i^\mathbf{MSR} - \eta = 0, \\
\eta \leq 0, \quad w_t \odot \eta = 0.
\]

Hence:

\[ c_t^R(w_t^\mathbf{MSR}) = -w_t \odot \eta = 0, \]

so that all contributions are equal. This shows that the MSR achieves risk parity for \( R_t \).
Appendices

B. Risk Contributions for Downside Risk Measures
This appendix gives expressions for the risk contributions associated with downside risk measures. For a risk measure $R_t$, these contributions are defined as:

$$c_t^R(w_t) = w_t \odot \frac{\partial R_t}{\partial w_t},$$

where $\frac{\partial R_t}{\partial w_t}$ is the $N \times 1$ gradient vector of $R_t$.

B.1 Gaussian Semi-volatility
From (A.2), we have (omitting the argument $w_t$ for brevity):

$$\text{GSV}^0_t = \sigma_P \sqrt{f(\ell_P)},$$

with $f(x) = (1 + x^2) \mathcal{N}(-x) - xn(x)$ and $\ell_P = \mu_P / \sigma_P$ and $\mu_P = w'_t \mu_t$. Hence:

$$\frac{\partial \text{GSV}^0_t}{\partial w_t} = \frac{\sigma_P}{2 \sqrt{f(\ell_P)}} \frac{\partial \sigma_P}{\partial w_t} + \frac{\sigma_P}{2 \sqrt{f(\ell_P)}} \frac{\partial \ell_P}{\partial w_t}$$

$$= \frac{\text{GSV}^0_t}{\sigma_P^2} \Sigma_t w_t + \frac{\sigma_P^2}{\text{GSV}^0_t} \left[ \ell_P \mathcal{N}(-\ell_P) - \frac{n(\ell_P)}{\sigma_P} \right] \times \left[ \frac{1}{\sigma_P} \mu_t - \frac{\mu_P}{\sigma_P^2} \Sigma_t w_t \right],$$

so that:

$$c_t^{\text{GSV}} = \frac{\text{GSV}^0_t}{\sigma_P^2} w_t \odot \Sigma_t w_t + \frac{\ell_P \mathcal{N}(-\ell_P) - \frac{n(\ell_P)}{\sigma_P}}{\text{GSV}^0_t} w_t \odot \left[ \frac{1}{\sigma_P} \mu_t - \frac{\mu_P}{\sigma_P^2} \Sigma_t w_t \right].$$

B.2 Gaussian Value-at-Risk
Expression (2.13) can be immediately differentiated with respect to $w_t$, which gives:

$$c_t^{\text{VaR}} = w_t \odot \left[ -\mu_t + \mathcal{N}^{-1}(\alpha) \frac{1}{\sigma_P} \Sigma_t w_t \right].$$

B.3 Non-Gaussian Value-at-Risk
The expressions for the partial derivatives $\partial \text{NGVaR}^\alpha_t / \partial w_{it}$ can be found in Boudt et al. (2008). We sketch their derivation here for the sake of completeness. By (2.14), we have:

$$\frac{\partial \text{NGVaR}^\alpha_t}{\partial w_t} = \frac{\partial \text{VaR}^\alpha_t}{\partial w_t} + \left[ \text{NGVaR}^\alpha_t - \text{VaR}^\alpha_t \right] \times \frac{1}{\sigma_P} \Sigma_t w_t$$

$$+ \sigma_P \left[ \frac{q^2 - 1}{6} \frac{\partial \text{Sk}_t}{\partial w_t} + \frac{q^3 - 3q}{24} \frac{\partial \text{Ku}_t}{\partial w_t} - \frac{2q^3 - 5q}{18} \frac{\partial \text{Sk}_t}{\partial w_t} \right].$$

Let us define $m_3 = E_t[(w'_t X_{t,t+h} - \mu_P)^3]$ and $m_4 = E_t[(w'_t X_{t,t+h} - \mu_P)^4]$. Hence, we have:

$$\frac{\partial \text{Sk}_t}{\partial w_t} = \frac{1}{\sigma_P^3} \frac{\partial m_3}{\partial w_t} - \frac{3m_3}{\sigma_P^4} \Sigma_t w_t,$$

$$\frac{\partial \text{Ku}_t}{\partial w_t} = \frac{1}{\sigma_P^4} \frac{\partial m_4}{\partial w_t} - \frac{4m_4}{\sigma_P^5} \Sigma_t w_t.$$
Appendices

Note that \( m_3 = w_t' M_{3t}(w_t \otimes w_t) \) and \( m_4 = w_t' M_{4t}(w_t \otimes w_t \otimes w_t) \). Hence:

\[
\frac{\partial m_3}{\partial w_t} = 3 M_{3t}(w_t \otimes w_t), \\
\frac{\partial m_4}{\partial w_t} = 4 M_{4t}(w_t \otimes w_t \otimes w_t).
\]

This completes the computation of the contributions to non-Gaussian VaR.

C. Analytical Computation of Risk Parity Portfolios In Special Cases

In this appendix, we provide sufficient conditions for the "exact" RP portfolio based on a risk measure (i.e., the one defined by \( (2.1) \)) to coincide with the portfolio in which weights are inversely proportional to expected returns. We consider a (homogenous) risk measure \( R_t \) which can be written as a function of portfolio expected return and volatility:

\[
R_t(w_t) = F(w_t' \mu_t, \sqrt{w_t' \Sigma_t w_t}),
\]

\( F \) being an homogenous function. The volatility itself, and the Gaussian semivolatility and VaR fit into this class. The non-Gaussian VaR does not, because it also depends on higher-order moments. Denoting with \( F_1 \) and \( F_2 \) the partial derivatives of \( F \), we have:

\[
c_t^R(w_t) = F_1 w_t \otimes \mu_t + \frac{F_2}{\sigma p_t} w_t \otimes \Sigma_t w_t.
\]

Assume that \( \mu_t = \frac{x}{w_t' \sigma_{p,t}^{vol}} \) for some scalar \( x \). Then:

\[
c_t^R(w_t, \sigma_{p,t}^{vol}) = \left[ F_1 x + \frac{F_2}{N} \right] 1,
\]

so that the volatility-RP portfolio also achieves risk parity with respect to \( R \). We then have \( w_t^{RP,R} = w_t^{RP,vol} \), and because \( w_t^{RP,vol} = \frac{x}{\mu_t} \), the weights are inversely proportional to expected returns. We can summarise this discussion as follows:

Consider a risk measure satisfying \( (C.1) \). If \( \mu_t \) is of the form \( \frac{x}{w_t' \sigma_{p,t}^{vol}} \) for some scalar \( x \), then we have \( w_t^{RP,R} = w_t^{RP,vol} \), and the weights are inversely proportional to expected returns.

In the case of two assets, the condition on \( \mu_t \) reduces to the condition that both assets have the same ratio \( \mu_{il}/\sigma_{il} \).

Assume now that \( \mu_{il} \) has the form \( \frac{x}{w_t' \sigma_{p,t}^{vol}} \), and that all assets have the same pairwise correlation. Then, by Proposition 1, the weights of the volatility-RP portfolio are of the form \( w_t^{RP,vol} = \frac{y}{\sigma} \), where \( \sigma \) is the \( N \times 1 \) vector of volatilities. Moreover, the previous discussion shows that \( w_t^{RP,R} = w_t^{RP,vol} \), hence \( w_t^{RP,R} = \frac{y}{\sigma} \). Let us now evaluate the risk measure \( R_t \) on each asset:

---

24 - For a \( N \times 1 \) vector \( y \) with non-zero elements, we use the notation \( x/y \) to denote the vector whose elements are \( x/y_1, \ldots, x/y_N \).
where the second equality uses the fact that $\mu_t = xy\sigma$ and the third one uses the homogeneity of $F$. As a consequence, the weights of the $R$-RP portfolio are inversely proportional to constituents’ risk measures. We thus have the following result:

Consider a risk measure satisfying (C.1), and assume that $\mu_t$ is of the form $\frac{\mu_t}{\sigma_t}$ for some scalar $x$, and that all assets have the same pairwise correlation. Then we have $w_{it,RP} = w_{it,\text{vol}}$, a portfolio whose weights are inversely proportional to expected returns, to volatilities and to constituents’ risk measures.

This proposition gives sufficient conditions for the weights of the $R$-RP strategy to coincide with the weights of a heuristic strategy that weights constituents by the inverses of their risk measures. Such a heuristic weighting scheme is the transposition to another risk measure of the inverse-volatility rule: it is computationally easier to implement than the true $R$-RP strategy in that it does not require the numerical solution of (2.3), but it does not achieve risk parity for any set of parameter values.

In the case of two assets, the condition on pairwise correlations is always satisfied, so the $R$-RP strategy is equivalent to the heuristic version whenever the risk measure has the form (C.1) and the two assets have the same ratio $\frac{\mu_{it}}{\sigma_{it}}$.
Appendices

D. Tables

Table 1: Descriptive statistics on data; Monthly dataset; January 1973 - December 2012.

(a) Means (annualised).

<table>
<thead>
<tr>
<th>3-M rate (lev.)</th>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>DP (lev.)</th>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.053</td>
<td>0.066</td>
<td>4.685</td>
<td>0.030</td>
<td>0.076</td>
<td>0.094</td>
</tr>
</tbody>
</table>

(b) Volatilities (annualised).

<table>
<thead>
<tr>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>DP (lev.)</th>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
<th>DP (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.031</td>
<td>0.880</td>
<td>0.013</td>
<td>0.053</td>
<td>0.158</td>
<td>0.015</td>
<td>0.006</td>
</tr>
</tbody>
</table>

(c) Sharpe ratios (annualised).

<table>
<thead>
<tr>
<th>Bond index</th>
<th>Stock index</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.444</td>
<td>0.200</td>
</tr>
</tbody>
</table>

(d) Skewnesses.

<table>
<thead>
<tr>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.335</td>
<td>-0.692</td>
<td>-0.485</td>
</tr>
</tbody>
</table>

(e) Kurtosis.

<table>
<thead>
<tr>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.685</td>
<td>2.508</td>
<td>9.897</td>
</tr>
</tbody>
</table>

(f) Correlations.

<table>
<thead>
<tr>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
<th>DP (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>-0.718</td>
<td>-0.017</td>
<td>1.000</td>
<td>-0.930</td>
<td>-0.116</td>
</tr>
<tr>
<td>0.013</td>
<td>0.004</td>
<td>0.003</td>
<td>1.000</td>
<td>-0.461</td>
<td>1.000</td>
</tr>
<tr>
<td>0.094</td>
<td>-0.082</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>0.034</td>
<td>0.038</td>
<td>-0.461</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>0.631</td>
<td>-0.052</td>
<td>-0.168</td>
<td>-0.883</td>
<td>0.189</td>
<td>-0.078</td>
</tr>
<tr>
<td>-0.051</td>
<td>-0.068</td>
<td>-0.198</td>
<td>-0.883</td>
<td>0.189</td>
<td>-0.078</td>
</tr>
</tbody>
</table>

This table contains summary statistics computed from monthly observations that cover the period January 1973 - December 2012. "lev." means that a variable is taken in level, "diff." that it is differentiated, and "ret." that the logarithmic return is extracted. Means and volatilities of returns and differences, as well as Sharpe ratios, are annualised. Kurtosis are in excess of the kurtosis of the reference normal distribution (equal to 3).

Table 2: Statistics on stock and bond volatilities; Monthly returns; January 1978 - December 2012

(a) Averages of volatility measures.

<table>
<thead>
<tr>
<th>VB-RW</th>
<th>VB-GARCH</th>
<th>VB-DUR-OBS</th>
<th>VB-DUR-APP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.053</td>
<td>0.054</td>
<td>0.072</td>
<td>0.070</td>
</tr>
</tbody>
</table>

(b) Correlations and volatilities of volatility measures.

<table>
<thead>
<tr>
<th>VB-RW</th>
<th>VB-GARCH</th>
<th>VB-DUR-OBS</th>
<th>VB-DUR-APP</th>
</tr>
</thead>
<tbody>
<tr>
<td>VB-RW</td>
<td>0.013</td>
<td>0.018</td>
<td>0.011</td>
</tr>
<tr>
<td>VB-GARCH</td>
<td>0.007</td>
<td>0.018</td>
<td>0.011</td>
</tr>
<tr>
<td>VB-DUR-OBS</td>
<td>-0.555</td>
<td>-0.432</td>
<td>0.011</td>
</tr>
<tr>
<td>VB-DUR-APP</td>
<td>-0.506</td>
<td>-0.424</td>
<td>0.005</td>
</tr>
</tbody>
</table>

The signification of abbreviations is as follows: "VB-RW": RW bond volatility; "VB-GARCH": GARCH bond volatility; "VB-DUR-OBS": duration-based volatility computed with observed duration; "VB-DUR-APP": duration-based volatility computed with approximate duration; "VS-RW": RW stock volatility; "VS-GARCH": GARCH stock volatility. In Panel (b), diagonal elements are volatilities and off-diagonal elements are correlations.
Table 3: Concordance and discordance rates for risk measures and weights of RP strategies invested in stock and bond; January 1978-December 2012.

Panel (a) shows the discordance rates for the following bond risk measures: "VB-RW": rolling-window volatility; "VB-GARCH": GARCH volatility; "VB-DUR-OBS": duration-based volatility computed with observed duration; "VB-DUR-APP": duration-based volatility computed with approximate duration; "GSVB": Gaussian semivolatility; "GVAR99B": Gaussian VaR at 99%; "NGVAR99B": Cornish-Fisher VaR at 99%. The discordance rate is defined as the percentage of months in the sample where the risk measure and the yield moved in opposite directions. Panel (b) shows the concordance rates between bond weight and bond yield for one unconditional risk parity strategy (URP-VOL-RW), two conditional risk parity (CRP) strategies based on volatility (CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP), three CRP strategies based on downside measures (CRP-GSV, CRP-GVAR99 and CRP-NGVAR99), and two maximum Sharpe ratio strategies (MSR-DIFF-SR and MSR-SAME-SR), which differ through the expected return estimates for the stock and the bond.

Table 4: Predictive regressions of asset returns; January 1973-December 2012.

These tables give the beta, the t-statistics in parenthesis, the $R^2$ and the adjusted $R^2$ for predictive regressions of returns: future stock returns are regressed on the current dividend-price ratio, and future bond returns on the current bond yield. The columns correspond to various horizons for future returns. Observations are monthly, so the returns are overlapping. t-statistics are corrected for autocorrelation using Newey-West adjustment (Newey and West, 1987), and the number of lags is selected according to a heuristic rule based on Newey and West (1994).
Appendices

Table 5: Bond weight in risk parity portfolios invested in stock and bond on December 31, 2012.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>URP-VOL-RW</td>
<td>0.798</td>
</tr>
<tr>
<td>CRP-VOL-GARCH</td>
<td>0.829</td>
</tr>
<tr>
<td>CRP-VOL-DUR-OBS</td>
<td>0.709</td>
</tr>
<tr>
<td>CRP-VOL-DUR-APP</td>
<td>0.712</td>
</tr>
<tr>
<td>CRP-GSV</td>
<td>0.708</td>
</tr>
<tr>
<td>CRP-GVAR99</td>
<td>0.700</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>0.533</td>
</tr>
<tr>
<td>MSR-SAME-SR</td>
<td>0.681</td>
</tr>
<tr>
<td>MSR-DIFF-SR</td>
<td>0.652</td>
</tr>
</tbody>
</table>

This table shows the weight allocated to the bond by the various risk parity strategies considered in this paper on December 31, 2012. All these strategies are invested in the stock and the bond. URP-VOL-RW is a RP strategy that uses rolling-window volatility estimates for both assets; CRP-VOL-GARCH is a RP strategy that uses GARCH(1,1) volatility estimates; CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP are two RP strategies where the stock volatility is estimated over a rolling window, and the bond volatility is proportional to a duration measure, either the observed duration or the approximate duration; CRP-GSV, CRP-GVAR99 and CRP-NGVAR99 are three RP strategies that equate the contributions to dissymmetric risk measures, namely the Gaussian semivolatility, the Gaussian VaR at 99%, and the Cornish-Fisher VaR at 99%; finally, MSR-DIFF-SR and MSR-SAME-SR are two MSR strategies, which differ through the priors that they use for expected returns.
Table 6: Comparison between actual data (January 1973-December 2012; upright font) and simulated data (December 2012-December 2027; italic font).

(a) Means (annualised).

<table>
<thead>
<tr>
<th>3-M rate (lev.)</th>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>DP (lev.)</th>
<th>Bond index (lev.)</th>
<th>Stock index (lev.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.053</td>
<td>0.066</td>
<td>4.685</td>
<td>0.030</td>
<td>0.076</td>
<td>0.094</td>
</tr>
<tr>
<td>0.045</td>
<td>0.057</td>
<td>4.762</td>
<td>0.026</td>
<td>0.058</td>
<td>0.105</td>
</tr>
</tbody>
</table>

(b) Volatilities (annualised).

<table>
<thead>
<tr>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>DP (lev.)</th>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
<th>DP (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.031</td>
<td>0.800</td>
<td>0.013</td>
<td>0.053</td>
<td>0.158</td>
<td>0.015</td>
<td>0.006</td>
</tr>
<tr>
<td>0.018</td>
<td>0.184</td>
<td>0.067</td>
<td>0.071</td>
<td>0.150</td>
<td>0.015</td>
<td>0.006</td>
</tr>
</tbody>
</table>

(c) Sharpe ratios (annualised).

<table>
<thead>
<tr>
<th>Bond index</th>
<th>Stock index</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.444</td>
<td>0.260</td>
</tr>
</tbody>
</table>

(d) Skewnesses.

<table>
<thead>
<tr>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.335</td>
<td>-0.692</td>
<td>-0.485</td>
</tr>
<tr>
<td>-0.067</td>
<td>0.005</td>
<td>-0.000</td>
</tr>
</tbody>
</table>

(e) Kurtosis.

<table>
<thead>
<tr>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.685</td>
<td>2.598</td>
<td>9.897</td>
</tr>
</tbody>
</table>

(f) Correlations.

<table>
<thead>
<tr>
<th>Bond yield (lev.)</th>
<th>Bond dur. (lev.)</th>
<th>Bond index (ret.)</th>
<th>Stock index (ret.)</th>
<th>Bond yield (diff.)</th>
<th>Bond dur. (diff.)</th>
<th>DP (diff.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>1.000</td>
<td>-0.718</td>
<td>1.000</td>
<td>-0.999</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>-0.909</td>
<td>1.000</td>
<td>0.056</td>
<td>-0.917</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.035</td>
<td>-0.038</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.013</td>
<td>0.004</td>
<td>0.093</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.016</td>
<td>-0.010</td>
<td>0.183</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.094</td>
<td>-0.082</td>
<td>-0.030</td>
<td>-0.116</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.044</td>
<td>-0.014</td>
<td>-0.966</td>
<td>-0.122</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.034</td>
<td>0.034</td>
<td>0.038</td>
<td>-0.461</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.034</td>
<td>0.034</td>
<td>0.038</td>
<td>-0.461</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.031</td>
<td>-0.052</td>
<td>-0.168</td>
<td>-0.883</td>
<td>0.180</td>
<td>-0.078</td>
<td>1.000</td>
</tr>
<tr>
<td>0.027</td>
<td>-0.037</td>
<td>-0.186</td>
<td>-0.893</td>
<td>0.188</td>
<td>-0.187</td>
<td>1.000</td>
</tr>
</tbody>
</table>

The numbers in upright font are summary statistics computed from monthly observations that cover the period January 1973 - December 2012 (these numbers are the same as in Table 1). The numbers in italic shape are the averages across simulated 2,000 paths of the same statistics. The simulations relate to the period December 2012 - December 2027. "lev." means that a variable is taken in level, "diff." that it is differentiated, and "ret." that the logarithmic return is extracted. Means and volatilities of returns and differences, as well as Sharpe ratios, are annualised. Kurtosis are in excess of the kurtosis of the reference normal distribution (equal to 3). Simulation parameters are described in Section 4.1.2.
## Appendices

### Table 7: Simulated performance statistics of risk parity and fixed-mix strategies invested in stock and bond; Monthly returns; December 2012-December 2027.

(a) Scenario 1 - Average interest rate and excess returns.

<table>
<thead>
<tr>
<th></th>
<th>2013</th>
<th>2014</th>
<th>2015</th>
<th>2016</th>
<th>2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (av.)</td>
<td>0.020</td>
<td>0.036</td>
<td>0.047</td>
<td>0.054</td>
<td>0.059</td>
</tr>
<tr>
<td>Bond index</td>
<td>-0.097</td>
<td>-0.053</td>
<td>-0.034</td>
<td>-0.015</td>
<td>-0.005</td>
</tr>
<tr>
<td>Stock index</td>
<td>0.061</td>
<td>0.062</td>
<td>0.054</td>
<td>0.059</td>
<td>0.061</td>
</tr>
<tr>
<td>AVG-URP-VOL-RW</td>
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<td>-0.020</td>
<td>-0.008</td>
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</tr>
<tr>
<td>URP-VOL-RW</td>
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<td>-0.021</td>
<td>-0.006</td>
<td>0.011</td>
<td>0.020</td>
</tr>
<tr>
<td>CRP-VOL-DUR-APP</td>
<td>-0.079</td>
<td>-0.037</td>
<td>-0.021</td>
<td>-0.004</td>
<td>0.005</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>-0.018</td>
<td>0.009</td>
<td>0.014</td>
<td>0.025</td>
<td>0.031</td>
</tr>
<tr>
<td>MSR-DIFF-SR</td>
<td>-0.082</td>
<td>-0.037</td>
<td>-0.021</td>
<td>-0.004</td>
<td>0.005</td>
</tr>
<tr>
<td>MSR-SAME-SR</td>
<td>-0.084</td>
<td>-0.039</td>
<td>-0.023</td>
<td>-0.005</td>
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</table>

(b) Scenario 1 - Volatilities.

<table>
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<tbody>
<tr>
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<tr>
<td>Bond index</td>
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<td>0.072</td>
<td>0.071</td>
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<tr>
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<tr>
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<td>0.069</td>
<td>0.068</td>
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</tr>
<tr>
<td>URP-VOL-RW</td>
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<td>0.070</td>
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<tr>
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<td>0.068</td>
<td>0.067</td>
<td>0.066</td>
<td>0.065</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
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<td>0.084</td>
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<tr>
<td>MSR-DIFF-SR</td>
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<td>0.070</td>
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<tr>
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(c) Scenario 2 - Average interest rate and excess returns.

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<td>0.055</td>
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<tr>
<td>Bond index</td>
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<td>0.010</td>
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<td>Stock index</td>
<td>0.061</td>
<td>0.007</td>
<td>-0.102</td>
<td>0.036</td>
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<tr>
<td>AVG-URP-VOL-RW</td>
<td>-0.117</td>
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<td>-0.028</td>
<td>0.020</td>
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<tr>
<td>URP-VOL-RW</td>
<td>-0.127</td>
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<td>-0.029</td>
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<td>0.039</td>
</tr>
<tr>
<td>CRP-VOL-DUR-APP</td>
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<td>-0.017</td>
<td>0.012</td>
<td>0.021</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>-0.064</td>
<td>-0.015</td>
<td>-0.045</td>
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<td>MSR-DIFF-SR</td>
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<td>0.015</td>
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<td>MSR-SAME-SR</td>
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(d) Scenario 2 - Volatilities.

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<th>2017</th>
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<tr>
<td>Yield</td>
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<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>Bond index</td>
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<td>0.069</td>
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<td>0.068</td>
</tr>
<tr>
<td>Stock index</td>
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<td>0.193</td>
<td>0.243</td>
<td>0.244</td>
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<td>0.077</td>
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<td>0.076</td>
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<tr>
<td>URP-VOL-RW</td>
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<td>0.087</td>
<td>0.076</td>
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<tr>
<td>CRP-VOL-DUR-APP</td>
<td>0.073</td>
<td>0.069</td>
<td>0.070</td>
<td>0.069</td>
<td>0.066</td>
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<tr>
<td>CRP-NGVAR99</td>
<td>0.085</td>
<td>0.105</td>
<td>0.123</td>
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<td>0.095</td>
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<td>MSR-DIFF-SR</td>
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<td>0.069</td>
<td>0.069</td>
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<td>MSR-SAME-SR</td>
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<td>0.069</td>
<td>0.069</td>
<td>0.068</td>
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</tr>
</tbody>
</table>

Panels (a) and (c) contain the annual averages of the bond yield in the first line, and the annual excess returns over the risk-free rate of the bond, the stock, and six portfolio strategies. Panels (b) and (d) contain the volatilities of monthly yield changes and monthly returns on the constituents and the six portfolios. All excess returns and volatilities are expressed in annual terms. The statistics are first computed along each of the 2,000 simulated paths, and are then averaged across paths. The definitions of the risk parity strategies and of the fixed-mix AVG-URP-VOL-RW are given in Section 4.3, and the parameter values that characterise each economic scenario are in Section 4.1.2.
Appendices

Table 8: Bond weight in portfolios invested in bond, stock and commodity indices on December 31, 2012.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Bond Weight</th>
</tr>
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<tbody>
<tr>
<td>URP-VOL-RW</td>
<td>0.776</td>
</tr>
<tr>
<td>CRP-VOL-GARCH</td>
<td>0.732</td>
</tr>
<tr>
<td>CRP-VOL-DUR-OBS</td>
<td>0.080</td>
</tr>
<tr>
<td>CRP-VOL-DUR-APP</td>
<td>0.684</td>
</tr>
<tr>
<td>CRP-GSV</td>
<td>0.683</td>
</tr>
<tr>
<td>CRP-GVAR99</td>
<td>0.683</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>0.590</td>
</tr>
<tr>
<td>MSR-SAME</td>
<td>0.695</td>
</tr>
<tr>
<td>MSR-DIFF</td>
<td>0.656</td>
</tr>
</tbody>
</table>

This table shows the weight allocated to the bond by the various Risk Parity strategies considered in this paper on December 31, 2012. All these strategies are invested in the stock and the bond. URP-VOL-RW is a RP strategy that uses rolling-window volatility estimates for both assets; CRP-VOL-GARCH is a RP strategy that uses GARCH(1,1) volatility estimates; CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP are two RP strategies where the stock volatility is estimated over a rolling window and the bond volatility is proportional to a duration measure, either the observed duration or the approximate duration; CRP-GSV, CRP-GVAR99 and CRP-NGVAR99 are three RP strategies that equate the contributions to dissymmetric risk measures, namely the Gaussian semivolatility, the Gaussian VaR at 99%, and the Cornish-Fisher VaR at 99%; finally, MSR-DIFF-SR and MSR-SAME-SR are two MSR strategies, which differ through the priors that they use for expected returns.


<table>
<thead>
<tr>
<th>Strategy</th>
<th>Corr. (w, Bond yield)</th>
<th>Concordant months (%) (Δw, ΔBond yield)</th>
</tr>
</thead>
<tbody>
<tr>
<td>URP-VOL-RW</td>
<td>0.620</td>
<td>50.239</td>
</tr>
<tr>
<td>CRP-VOL-GARCH</td>
<td>0.597</td>
<td>47.852</td>
</tr>
<tr>
<td>CRP-VOL-DUR-OBS</td>
<td>0.087</td>
<td>63.604</td>
</tr>
<tr>
<td>CRP-VOL-DUR-APP</td>
<td>0.152</td>
<td>68.616</td>
</tr>
<tr>
<td>CRP-GSV</td>
<td>0.096</td>
<td>75.098</td>
</tr>
<tr>
<td>CRP-GVAR99</td>
<td>0.016</td>
<td>73.866</td>
</tr>
<tr>
<td>CRP-NGVAR99</td>
<td>0.020</td>
<td>80.549</td>
</tr>
<tr>
<td>MSR-SAME</td>
<td>0.582</td>
<td>76.969</td>
</tr>
<tr>
<td>MSR-DIFF</td>
<td>0.612</td>
<td>80.072</td>
</tr>
</tbody>
</table>

This table reports the percentage of months in which the weight of bond in maximum Sharpe ratio strategies moved in the same direction as the yield. In strategy MSR-SAME-SR, the prior expected returns for the stock and the bond imply the same constant Sharpe ratio, which is the long-term Sharpe ratio of bonds (0.26), while in strategy MSR-DIFF-SR, these priors imply constant Sharpe ratios which are equal to the long-term Sharpe ratios of the asset classes (0.26 for the bond, 0.41 for the stock and 0.22 for the commodity index).
Appendices

Figure 1: Estimated bond and stock volatilities; Monthly returns; January 1973-December 2012.

The signification of abbreviations is as follows: "VB-RW": RW bond volatility; "VB-GARCH": GARCH bond volatility; "VB-DUR-OBS": duration-based volatility computed with observed duration; "VB-DUR-APP": duration-based volatility computed with approximate duration; "VS-RW": RW stock volatility; "VS-GARCH": GARCH stock volatility. All volatilities are annualised.
Figure 2: Weights of volatility-RP strategies invested in bond and stock; January 1978-December 2012.

This figure shows the weights of four bond-stock risk parity strategies that differ through the volatility estimates. In the strategy URP-VOL-RW, volatilities are estimated over a five-year rolling window. In CRP-VOL-GARCH, they are estimated with a GARCH(1,1) model. In CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP, stock volatility is equal to its rolling-window estimate, but bond volatility is taken proportional to a duration measure. This measure is the observed duration in CRP-VOL-DUR-OBS, and the approximate duration (2.10) in CRP-VOL-DUR-APP.
Appendices

Figure 3: Estimated stock and bond expected returns; January 1978-December 2012.

This figure shows estimated stock and bond expected returns. The “forecast” is the fitted value in a predictive regression at the one-year horizon (the predictor is the dividend-price ratio for the stock and the yield for the bond). The “prior” is the expected return that is consistent with a given target Sharpe ratio, taken equal to the long-term value. The “posterior” is a convex combination of the forecast and the prior, where the weights are inversely proportional to the variances of two estimators.
Figure 4: Downside risk measures for stock and bond; January 1978-December 2012.

This figure shows the values of three downside risk measures for the stock (S) and the bond (B). Stock volatility is the RW volatility, and bond volatility is the DUR-APP estimate (volatility proportional to approximate duration). Expected return estimates are obtained by shrinking a forecast coming from a linear regression (see Table 4) towards a prior that reflects the Sharpe ratios over the period 1926-2012. For the non-Gaussian VaR, the skewnesses are set to zero for robustification purposes, stock kurtosis is estimated as its sample counterpart, and bond kurtosis is estimated as sample yield kurtosis.
Appendices

Figure 5: Weights of RP strategies based on downside risk measures and invested in bond and stock; January 1978-December 2012.

Each CRP strategy equates the contributions of the stock and the bond to a downside risk measure of the portfolio: Gaussian semivolatility (CRP-GSV), Gaussian VaR at 99% (CRP-GVAR99) and non-Gaussian VaR at 99% (CRP-NGVAR99). The weights of each strategy have been obtained by solving the system (2.3). Expected returns of the stock and the bond are estimated by shrinking a forecast issued from a predictive regression towards a prior. This prior implies a constant Sharpe ratio, which is the long-term Sharpe ratio of the asset class (0.26 for the bond and 0.41 for the stock). The predictors of future returns are the dividend-price ratio for the stock and the yield-to-redemption for the bond. Stock volatility is estimated as the RW volatility, and bond volatility is estimated as the DUR-APP volatility (a volatility proportional to duration).

Figure 6: Weights of MSR strategies invested in bond and stock; January 1978-December 2012.

The weights of the maximum Sharpe ratio portfolios have been obtained by solving the optimization program (2.16). Expected returns of the stock and the bond are estimated by shrinking a forecast issued from a predictive regression towards a prior. For strategy MSR-SAME-SR, this prior implies a constant Sharpe ratio, which is taken to be the same for both assets, namely the long-term Sharpe ratio of the bond (0.26). For strategy MSRDIFF-SR, this prior implies a constant Sharpe ratio of 0.41 for the stock and 0.26 for the bond. The predictors of future returns used in the forecasts are the dividend-price ratio for the stock and the yield-to-redemption for the bond. Stock volatility is estimated as the RW volatility, and bond volatility is estimated as the DUR-APP volatility (a volatility proportional to duration).
Appendices

Figure 7: Bond yield and duration in historical and simulated datasets; January 1973-December 2027.

The curves on the left side of the vertical dashed line show the observed yield and the approximate duration (computed as (2.10)) for the January 1973-December 2012. The curves on the right side are the averages across 2,000 paths of the simulated yield and duration. In scenario 1, interest rates grow progressively from their level on Dec. 31, 2012 to their long-term mean, and in scenario 2, they increase rapidly. Simulation parameters are described in Section 4.1.2.
Appendices

Figure 8: Bond and stock risk measures in historical dataset and scenario 1; January 1973-December 2027.

Panel (a) shows the rolling-window volatilities of the stock and the bond computed from realised returns over the period January 1973-December 2012, and the expected rolling-window volatilities computed from simulated returns over December 2012-December 2027. The vertical lines identify December 2012. Also plotted is the DUR-APP volatility, which is a bond volatility measure proportional to the duration. Expected values are computed as the averages of 2,000 simulated values, and volatilities are annualised. Panel (b) shows the non-Gaussian Value-at-Risk at 99%. Simulation parameters relate to the scenario 1 described in Section 4.1.2.
Appendices

Figure 9: Estimated commodity volatilities; Monthly returns; January 1973-December 2012.

The signification of abbreviations is as follows: “VC-RW”: volatility estimated over a five-year rolling window; “VC-GARCH”: volatility estimated from a GARCH(1,1) model. All volatilities are annualised.

Figure 10: Estimated pairwise correlations between assets; Monthly returns; January 1973-December 2012.

Correlations are estimated over a five-year rolling window with monthly observations. The signification of abbreviations is as follows: “CORR-BC-RW”: bond-commodity correlation; “CORR-BS-RW”: bond-stock correlation; “CORR-SC-RW”: stock-commodity correlation.
Appendices

Figure 11: Weights of volatility-RP strategies invested in bond, stock and commodity indices; January 1978–December 2012.

This figure shows the weights of four risk parity portfolios invested in stocks, bonds and commodities, which differ through the volatility estimates. In the strategy URP-VOL-RW, volatilities are estimated over a five-year rolling window. In CRP-VOL-GARCH, they are estimated with a GARCH(1,1) model. In CRP-VOL-DUR-OBS and CRP-VOL-DUR-APP, stock and commodity volatilities are equal to their rolling-window estimates, but bond volatility is taken proportional to a duration measure. This measure is the observed duration in CRP-VOL-DUR-OBS, and the approximate duration (2.10) in CRP-VOL-DUR-APP. The correlations are the rolling-window estimates shown in Figure 10.
Appendices

Figure 12: Weights of RP strategies based on downside risk measures and invested in bond, stock and commodity indices; January 1978-December 2012.

Each CRP strategy equates the contributions of the bond, the stock and the commodity indices to a downside risk measure of the portfolio: Gaussian semivolatility (CRP-GSV), Gaussian VaR at 99% (CRP-GVAR99) and non-Gaussian VaR at 99% (CRP-NGVAR99). The weights of each strategy have been obtained by solving the system (2.3). Expected returns of the stock and the bond are estimated by shrinking a forecast issued from a predictive regression towards a prior, whilst for the commodity they are considered as given by the prior itself (total shrinkage). The priors imply a constant Sharpe ratio, which is the long-term Sharpe ratio of the asset class (0.26 for the bond, 0.41 for the stock and 0.22 for the commodity). The predictors of future returns are the dividend-price ratio for the stock and the yield-to-redemption for the bond. Stock and commodity volatilities are estimated as the RW volatilities, and bond volatility is estimated as the DUR-APP volatility (a volatility proportional to duration).
Appendices

Figure 13: Weights of MSR strategies invested in bond, stock and commodity indices; January 1978–December 2012.

The weights of the maximum Sharpe ratio portfolios have been obtained by solving the optimization program (2.16). Expected returns of the stock and the bond are estimated by shrinking a forecast issued from a predictive regression towards a prior, whilst for the commodity they are considered as given by the prior itself (total shrinkage). For strategy MSR-SAME-SR, this prior implies a constant Sharpe ratio, which is taken to be the same for all the assets, namely the long-term Sharpe ratio of the bond (0.26). For strategy MSR-DIFF-SR, this prior implies a constant Sharpe ratio of 0.26 for the bond, 0.41 for the stock and 0.22 for the commodity. The predictors of future returns used in the forecasts are the dividend-price ratio for the stock and the yield-to-redemption for the bond. Stock and commodity volatilities are estimated as the RW volatilities, bond volatility is instead estimated as the DUR-APP volatility (a volatility proportional to duration).
Appendices
References
References


References

References


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About EDHEC-Risk Institute

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- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

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• Advanced Modelling for Alternative Investments, in partnership with Newedge Prime Brokerage
• Advanced Investment Solutions for Liability Hedging for Inflation Risk, in partnership with Ontario Teachers’ Pension Plan
• The Case for Inflation-Linked Corporate Bonds: Issuers’ and Investors’ Perspectives, in partnership with Rothschild & Cie
• Solvency II, in partnership with Russell Investments
• Structured Equity Investment Strategies for Long-Term Asian Investors, in partnership with Société Générale Corporate & Investment Banking

The philosophy of the Institute is to validate its work by publication in international academic journals, as well as to make it available to the sector through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises three conferences for professionals in order to present the results of its research, one in London (EDHEC-Risk Days Europe), one in Singapore (EDHEC-Risk Days Asia), and one in New York (EDHEC-Risk Days North America) attracting more than 2,500 professional delegates.

EDHEC also provides professionals with access to its website, www.edhec-risk.com, which is entirely devoted to international asset management research. The website, which has more than 58,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC’s analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 1.5 million readers.

EDHEC-Risk Institute: Key Figures, 2011-2012

<table>
<thead>
<tr>
<th>Nbr of permanent staff</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nbr of research associates</td>
<td>20</td>
</tr>
<tr>
<td>Nbr of affiliate professors</td>
<td>28</td>
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<tr>
<td>Overall budget</td>
<td>€13,000,000</td>
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<tr>
<td>External financing</td>
<td>€5,250,000</td>
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<tr>
<td>Nbr of conference delegates</td>
<td>1,860</td>
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<tr>
<td>Nbr of participants at research seminars</td>
<td>640</td>
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<tr>
<td>Nbr of participants at EDHEC-Risk Institute Executive Education seminars</td>
<td>182</td>
</tr>
</tbody>
</table>
About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance
The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world’s best universities, such as Princeton, Wharton, Oxford, Chicago and CalTech, and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

Research for Business
The Institute’s activities have also given rise to executive education and research service offshoots. EDHEC-Risk’s executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes. In partnership with CFA Institute, it has developed advanced seminars based on its research which are available to CFA charterholders and have been taking place since 2008 in New York, Singapore and London.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

As part of its policy of transferring know-how to the industry, EDHEC-Risk Institute has also set up ERI Scientific Beta. ERI Scientific Beta is an original initiative which aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.
EDHEC-Risk Institute
Publications and Position Papers
(2011-2014)

2014
• Amenc, N., and F. Ducoulombier. Index Transparency – A Survey of European Investors Perceptions, Needs and Expectations (March).
• Ducoulombier, F., F. Goltz, V. Le Sourd, and A. Lodh. The EDHEC European ETF Survey 2013 (March).
• Badaoui, S., Deguest, R., L. Martellini and V. Milhau. Dynamic Liability-Driven Investing Strategies: The Emergence of a New Investment Paradigm for Pension Funds? (February).
• Deguest, R., and L. Martellini. Improved Risk Reporting with Factor-Based Diversification Measures (February).

2013
• Martellini, L., and V. Milhau. Analysing and decomposing the sources of added-value of corporate bonds within institutional investors' portfolios (August).
• Deguest, R., L. Martellini, and A. Meucci. Risk parity and beyond - From asset allocation to risk allocation decisions (June).
• Blanc-Brude, F., Cocquemas, F., Georgieva, A. Investment Solutions for East Asia's Pension Savings - Financing lifecycle deficits today and tomorrow (May)
• Blanc-Brude, F. and O.R.H. Ismail. Who is afraid of construction risk? (March)
• Lixia, L., L. Martellini, and S. Stoyanov. The relevance of country- and sector-specific model-free volatility indicators (March).
• Deguest, R., L. Martellini, and V. Milhau. The benefits of sovereign, municipal and corporate inflation-linked bonds in long-term investment decisions (February).
• Deguest, R., L. Martellini, and V. Milhau. Hedging versus insurance: Long-horizon investing with short-term constraints (February).
• Padmanaban, N., M. Mukai, L. Tang, and V. Le Sourd. Assessing the quality of asian stock market indices (February).


• Cocquemas, F. Towards better consideration of pension liabilities in European Union countries (January).

• Blanc-Brude, F. Towards efficient benchmarks for infrastructure equity investments (January).

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• Arias, L., P. Foulquier and A. Le Maistre. Les impacts de Solvabilité II sur la gestion obligataire (December).

• Arias, L., P. Foulquier and A. Le Maistre. The Impact of Solvency II on Bond Management (December).

• Amenc, N., and F. Ducoulombier. Proposals for better management of non-financial risks within the European fund management industry (December).

• Cocquemas, F. Improving Risk Management in DC and Hybrid Pension Plans (November).


• La gestion indicielle dans l’immobilier et l’indice EDHEC IEIF Immobilier d’Entreprise France (September).

• Real estate indexing and the EDHEC IEIF commercial property (France) index (September).

• Goltz, F., S. Stoyanov. The risks of volatility ETNs: A recent incident and underlying issues (September).

• Almeida, C., and R. Garcia. Robust assessment of hedge fund performance through nonparametric discounting (June).

• Amenc, N., F. Goltz, V. Milhau, and M. Mukai. Reactions to the EDHEC study “Optimal design of corporate market debt programmes in the presence of interest-rate and inflation risks” (May).

• Goltz, F., L. Martellini, and S. Stoyanov. EDHEC-Risk equity volatility index: Methodology (May).


EDHEC-Risk Institute Publications
(2011–2014)

- Schoeffler, P. Optimal market estimates of French office property performance (March).
- Le Sourd, V. Performance of socially responsible investment funds against an efficient SRI Index: The impact of benchmark choice when evaluating active managers – an update (March).
- Sender, S. Shifting towards hybrid pension systems: A European perspective (March).
- Blanc-Brude, F. Pension fund investment in social infrastructure (February).
- Schoeffler, P. Les estimateurs de marché optimaux de la performance de l’immobilier de bureaux en France (January).

2011
- Deguest, R., Martellini, L., and V. Milhau. Life-cycle investing in private wealth management (October).
- Le Sourd, V. Performance of socially responsible investment funds against an Efficient SRI Index: The Impact of Benchmark Choice when Evaluating Active Managers (September).

• Charbit, E., Giraud J. R., F. Goltz, and L. Tang Capturing the market, value, or momentum premium with downside Risk Control: Dynamic Allocation strategies with exchange-traded funds (July).
• Scherer, B. An integrated approach to sovereign wealth risk management (June).
• Martellini, L., and V. Milhau. Capital structure choices, pension fund allocation decisions, and the rational pricing of liability streams (June).
• Sender, S. The elephant in the room: Accounting and sponsor risks in corporate pension plans (March).
• Martellini, L., and V. Milhau. Optimal design of corporate market debt programmes in the presence of interest-rate and inflation risks (February).

2012
• Till, H. Who sank the boat? (June).

2011
• Amenc, N., and S. Sender. Response to ESMA consultation paper to implementing measures for the AIFMD (September).
• Uppal, R. A Short note on the Tobin Tax: The costs and benefits of a tax on financial transactions (July).
• Till, H. A review of the G20 meeting on agriculture: Addressing price volatility in the food markets (July).
Notes