

Option Pricing and Hedging in the Presence of Basis Risk

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Abstract

This paper addresses the problem of option hedging and pricing when a futures contract, written either on the underlying asset or on some imperfectly correlated substitute for the underlying asset, is used in the dynamic replication of the option payoff. In the presence of unspanned basis risk modeled as a Brownian bridge process, which explicitly accounts for the convergence of the basis to zero as the futures contract approaches maturity, we are able to obtain an analytical expression for the optimal hedging strategy and corresponding option price. Empirical analysis suggests that the hedging demand against basis risk is an important ingredient of the hedging strategy. For reasonable parameter values, we also find the replication error implied by the optimal strategy to be substantially lower than that implied by heuristic strategies routinely used in practice.

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Because of the presence of higher liquidity and lower frictions, sellers of equity derivatives routinely hedge index options or basket options by trading in equity index futures as substitutes for trading in the cash underlying portfolios. Similarly, interest rate futures are typically used in dynamic hedging/replicating strategies for options on bonds and other fixed-income derivatives. If changes in the price of the futures and cash contracts were perfectly correlated, no further risk would be introduced, and one could perfectly offset any gain or loss in the option position by dynamically trading in the related futures contract. In general, however, some form of dynamic market incompleteness arises because the presence of *basis risk*, *i.e.*, the risk of unexpected deviations between the prices of the futures contract (*e.g.*, S&P 500 futures contract) and underlying cash instrument (*e.g.*, S&P 500 index). Beside, in situations where the futures contract used in the dynamic hedging strategy is written on some imperfect substitute for the underlying asset of the option (*e.g.*, when S&P 500 futures contracts are used to hedge options written on, say, the S&P 100 index, or some other basket of stocks extracted from the S&P 500 universe), market incompleteness becomes even more severe because of the additional presence of *cross-hedge risk*, *i.e.*, the risk induced by the imperfect correlation between the underlying asset for the option contract to be replicated (*e.g.*, S&P 100 index) and the underlying asset for the futures contract involved in the dynamic replication strategy (*e.g.*, the S&P 500 index).

In the presence of basis risk and/or cross-hedge risk it is impossible to obtain a price and replicating strategy for a contingent claim by arbitrage considerations only. In such incomplete markets, because there are infinitely many equivalent martingale measures (EMMs) consistent with the absence of arbitrage, one may simply obtain bounds on the price of a given security (see Detemple and Sundaresan (1999)).¹ In order to specify a unique dynamic hedging strategy and a unique corresponding price for the option, the pricing/hedging problem has to be embedded within the agent's portfolio decision problem (see Dybvig (1992) or Collin-Dufresne and Hugonnier (2007)). Roughly speaking, the objective for the seller of the option is first to find trading strategies with a terminal value that maximizes her expected utility, in the presence and in the absence of a short position in the option contract, respectively. Then the corresponding reservation price for the option is obtained as the initial investment that would make the agent indifferent in terms of indirect utility achieved with and without taking on the short position in the contingent claim.²

This paper addresses the problem of finding the reservation price, and corresponding hedging strategy, for an option position by an agent endowed with CARA preferences in an incomplete market setting when a futures contract written on the underlying asset, or on some imperfectly correlated substitute for the underlying asset, is used in the dynamic replication of the option payoff. As such, our paper complements and extends to non-linear payoffs previous work on optimal hedging of linear risk exposures with futures contracts (see Duffie and Richardson (1991) or Lioui and Poncet (2000)). We use a Brownian bridge process to model the dynamics of the log difference between the spot and the futures prices while explicitly accounting for one key feature of the problem, namely, the convergence of the basis to zero as the futures contract approaches maturity. The presence of basis risk involves a severe increase in complexity in the option pricing and hedging problem since it leads to the introduction of one additional state variable whose fluctuations are not spanned by existing securities. Despite this increased complexity, we are able to provide an explicit characterization of the option price and corresponding hedging strategy, for which we obtain analytical expressions in the limit of a vanishing risk aversion.³ In an empirical exercise where we calibrate the model to S&P500 index and futures prices, the hedging demand against basis risk is found to be an important ingredient of the underlying hedging strategy. We also find the replication error to be substantially smaller than that implied by the heuristic

1 - In most incomplete market situations, the no-arbitrage bounds are too wide to be of any practical use (see Soner *et al.* (1995)) for a proof that initially buying a share of the underlying stock is the cheapest dominating policy for option replication in the presence of transaction costs). In this context, Bernardo and Ledoit (2000) and Cochrane and Saa-Requejo (2000) propose to derive tighter bounds on options on non-traded assets by imposing a restriction more stringent than the mere absence of arbitrage. In particular, Cochrane and Saa-Requejo (2000) rule out "good deals", defined as portfolios with exceedingly high Sharpe ratios, while Bernardo and Ledoit (2000) rule out "quasi-arbitrage" opportunities, *i.e.*, portfolios with a gain-loss ratio above a certain value. While tighter than *no arbitrage* bounds, these *no good deal* bounds, however, cannot be used to identify a unique hedging strategy.

2 - Alternatively, one may introduce some heuristic optimality criterion, typically related to minimization of the replication error, that the strategy should satisfy (see Föllmer and Schweizer (1990) and Schweizer (1991) for an analysis of *global* and *local* risk-minimizing strategies in incomplete markets).

3 - For a general risk-aversion parameter value, the option price and hedging strategy can be obtained through standard numerical techniques.

strategy based on the Black (1976) futures option pricing formula, routinely used in practice. Interestingly, the superiority of the optimal strategy, which depends upon expected return estimates for both the underlying asset and the substitute, is found to be extremely robust with respect to estimation errors in such estimates. This is an attractive result given the notorious lack of robustness of expected return estimates. We also consider the more general situation where the underlying asset for the option contract to be replicated is different from, and imperfectly correlated with, the underlying asset for the futures contract involved in the dynamic replication strategy. In this context involving the simultaneous presence of basis risk and cross-hedge risk, we are able to maintain (for a vanishing risk-aversion) analytical expressions for the optimal hedging strategy and corresponding option price, which then involves three state variables, namely, the price of the underlying asset for the option, the price of the underlying asset of the futures contract, and the price of the futures contract (or equivalently the basis level).⁴

The rest of the paper is organized as follows. Section 1 introduces the model and derives the optimal hedging strategy and price for a contingent claim in the presence of basis risk. Section 2 presents a numerical assessment of the replication error implied by the optimal strategy, which we compare to the Black strategy. In section 3, we show how the model can be extended to accommodate cross-hedge risk in addition to basis risk. Technical details and proofs of the main results can be found in a dedicated appendix.⁵

1. Option Hedging Strategies in the Presence of Basis Risk

In this section, we introduce the model and we derive the optimal dynamic replicating strategy and corresponding price for a European call option in the presence of basis risk. The analysis could be straightforwardly extended to any European contingent claim.

1.1 Economic Framework

Uncertainty in the economy is represented through a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$ supporting two standard Brownian motions z^X and z^D with correlation coefficient ρ . The information set available to the investor at time t is given by \mathcal{F}_t , where the family $(\mathcal{F}_t)_{t \geq 0}$ is the augmented version of the filtration generated by z^X and z^D .⁶

We assume that some underlying asset (e.g., an equity index) follows a geometric Brownian motion under the historical probability measure \mathbb{P} :

$$dX_t = X_t [\mu_X dt + \sigma_X dz_t^X] \quad (1)$$

and the problem is to find an optimal dynamic hedging strategy for a standard European call option with maturity T written on this underlying asset, with a payoff $(X_T - K)^+$, when the asset X is *not* available for trading, and a futures contract on this underlying asset is used as a substitute for hedging purposes.

We let F_t be the futures price of the asset, which is the price at which the buyer and the seller of the contract agree at settlement date t for delivery at time T_0 . If the financial market were complete, *i.e.*, if dX and dF were perfectly correlated, the absence of arbitrage would imply the existence of a *unique* equivalent martingale measure (EMM), under which the discounted value of X would be a martingale, and under which the futures price F_t would be obtained as the expected value of X_T conditional upon the information available at date t (see Duffie 2001). In general, the presence of basis risk, *i.e.*, the risk of unexpected deviations between the prices of the futures contract and underlying cash instrument induces some form of dynamic

4 - A number of papers have considered the option pricing problem when the (non-traded) underlying asset and a (traded) substitute follow correlated geometric Brownian motions (see Hodges and Neuberger (1989), Rouge and El Karoui (2000), Zariphopoulou (2001), Henderson (2002), Musiela and Zariphopoulou (2004), Tehranchi (2004), Monoyios (2004) or Davis (2006), among others). Such modeling assumptions are inconsistent with the almost sure convergence of the basis to zero as the futures contract approaches maturity, and as such their results do not apply to the problem at hand. In other words, these papers have focused on cross-hedge risk, while our main focus is on basis risk.

5 - For the sake of brevity, some proofs have been omitted in the paper. They can be obtained from the authors upon request.

6 - In our model, this is equivalent to assuming that the investor observes both spot and futures prices.

incompleteness, which implies the existence of an infinite number of equivalent martingale measures consistent with the absence of arbitrage. As a result, the futures price process will have to be exogenously given, as opposed to endogenously obtained by no-arbitrage arguments. Our modeling choices regarding the dynamics of changes in the futures price are constrained, however, by the necessary convergence of the futures price towards the underlying asset price when approaching the futures contract maturity date. In other words, it must be the case that $F_{T_0} = X_{T_0}$ almost surely so as to rule out arbitrage opportunities. To account for the presence of this constraint, we propose to model the log difference $D_t \equiv \ln(F_t/X_t)$ as a *Brownian bridge* process (see Liu and Longstaff (2004) for an example of use of the Brownian bridge model in a portfolio choice context):⁷

$$dD_t = \frac{-aD_t}{T_0 - t} dt + \sigma_D dz_t^D \quad (2)$$

Hence, as t gets close to the maturity date T_0 , the drift for this process goes to negative infinity if D_t is positive, and to positive infinity if D_t is negative, which eventually forces the value of D at time T_0 to be zero almost surely. One can easily show that D_t admits a closed-form expression given by:

$$D_t = \left(\frac{T_0 - t}{T_0}\right)^a D_0 + \sigma_D (T_0 - t)^a \int_0^t \frac{1}{(T_0 - s)^a} dz_s^D$$

Note that this model encompasses the situation where X is traded, which is recovered as a special case of our model (2). Indeed, in this special case, we have (for a constant interest rate r) the textbook relation $F_t = e^{r(T_0-t)} X_t$, so that $D_t = r(T_0 - t)$, which implies a dynamics of the form (2) with $\sigma_D = 0$ and $a = 1$. Here, the risk-free rate is assumed to be constant for simplicity.⁸

As can easily be seen by applying Ito's lemma, the above model implies that the futures price evolves as:

$$dF_t = F_t[\mu_F dt + \sigma_F dz_t^F] \quad (3)$$

where z^F is a Brownian motion under \mathbb{P} , and σ_F and μ_F are the volatility and the expected return on the futures contract, respectively. These quantities are given by:

$$\begin{aligned} \sigma_F &= \sqrt{\sigma_X^2 + \sigma_D^2 + 2\rho\sigma_X\sigma_D} \\ \mu_F &= \mu_X - \frac{aD_t}{T_0 - t} + \frac{1}{2} [\sigma_D^2 + 2\rho\sigma_X\sigma_D] \\ dz_t^F &= \frac{\sigma_X dz_t^X + \sigma_D dz_t^D}{\sigma_F} \end{aligned} \quad (4)$$

It should be noted that the drift μ_F is stochastic unless D_t is deterministic. Besides, we obtain that the correlation between unexpected changes in the futures price and the spot price of the underlying asset is constant and given by:

$$\rho_{FX} = \frac{\sigma_X + \rho\sigma_D}{\sqrt{\sigma_X^2 + \sigma_D^2 + 2\rho\sigma_X\sigma_D}} = \frac{\sigma_X + \rho\sigma_D}{\sigma_F} \quad (5)$$

This correlation is equal to -1 or +1 if and only if σ_D is zero or ρ is equal to -1 or +1. That is, the futures and the spot prices are perfectly correlated or anti-correlated if and only if there is no basis risk ($\sigma_D = 0$) or if basis risk is spanned by uncertainty over returns on X ($|\rho| = 1$). The intuition (which is confirmed in the empirical calibration in section 2) suggests that changes in the futures price are positively correlated with changes in the spot price, *i.e.*, $\rho_{FX} > 0$.

7 - Postulating a Brownian bridge process for the log difference in prices of the futures and spot contracts, as opposed to the difference in prices, allows us to ensure that the futures price F remains positive at all times with probability 1.

8 - An extension to stochastic interest rates would be relatively straightforward. The optimal strategy in this context would formally resemble the one obtained with constant interest rates, but it would also involve an additional hedging demand with respect to interest rate uncertainty.

Of course, if the option and the future happen to have the same maturity date (*i.e.*, if $T_0 = T$), the problem of replicating the option has a well-known solution: since $F_T = X_T$ almost surely in this case, we have $(X_T - K)^+ = (F_T - K)^+$, and the option can be priced by arbitrage arguments to yield what is known as the Black formula (Black, 1976):⁹

$$c^B(t, F_t, X_t) = e^{-r(T-t)} [F_t \mathcal{N}(d_{1,t}^B) - K \mathcal{N}(d_{2,t}^B)] \quad (6)$$

where $d_{1,t}^B = \frac{1}{\sigma_F \sqrt{T-t}} [\ln \frac{F_t}{K} + \frac{1}{2} \sigma_F^2 (T-t)]$ and $d_{2,t}^B = d_{1,t}^B - \sigma_F \sqrt{T-t}$. The number of futures contracts to be held at time t is:

$$n_t^B = c_F^B(t, F_t, X_t) = e^{-r(T-t)} \mathcal{N}(d_{1,t}^B) \quad (7)$$

In practice, however, it is in general impossible to find futures contracts with a maturity date exactly matching the option contract maturity date, especially for over-the-counter options. Option traders therefore have in general to use futures contracts with a maturity date either strictly shorter than, or strictly longer than, the option maturity date. With no loss of generality, we focus on the latter situation in what follows, that is we consider a setting where the maturity T_0 for the futures contract is strictly greater than the maturity of the option contract T . Indeed, when the maturity of the futures contract is shorter than the maturity of the option, the futures position must be rolled-over. In the end, unless the option maturity date happens to coincide exactly with the maturity date of the last futures contract involved in the roll-over strategy, the problem boils down to a situation such that $T_0 > T$. In fact, we show in the empirical analysis in section 2 that even reasonably small differences between the option contract maturity date and the futures contract maturity date may result in large replication errors, especially when the optimal hedging strategy presented below is not used.

Finally, we also assume that $\sigma_D > 0$. If $\sigma_D = 0$, there is no basis risk, and D is a deterministic process, $D_t = D_0 \left(\frac{T_0-t}{T_0}\right)^a$. In this case, $X_T = F_T e^{-D_T}$ and the option payoff can be rewritten as $(X_T - K)^+ = e^{-D_T} (F_T - K e^{D_T})^+$. Thus, a long position in one call option on X is equivalent to a long position in e^{-D_T} call options on F , with a strike price $K e^{D_T}$, and the option can be again valued by the Black formula, and perfectly replicated by a self-financing strategy.

In the general incomplete market setting when $T_0 > T$ and $\sigma_D > 0$, the use of standard no-arbitrage arguments is not sufficient for characterizing the option price and related hedging strategy. In particular, the absence of arbitrage implies the existence of an infinite number of EMMs, and one has to embed the hedging/pricing problem into the agent's portfolio decision to narrow down to one the number of admissible EMMs. This is the focus of the next subsection.

1.2 Expected Utility Maximization and Option Replication in the Presence of Basis Risk

As explained in the introduction, option pricing problems in incomplete markets can be analyzed by comparing the maximal utility achieved with and without taking on the short position in the contingent claim, and defining the (selling) price for the option as the initial investment that would make the seller indifferent between these two strategies. Formally, we let n_t be the number of futures contracts held by the seller of the option at time t and V_t be the value of a portfolio strategy based on investing in cash and trading in the futures contract in a self-financing way, starting from the initial capital V_0 .¹⁰ The budget equation is thus given by:

$$dV_t = rV_t dt + n_t dF_t \quad (8)$$

The term $n_t dF_t$ on the right-hand side of (8) accounts for the presence of continuous margin calls from time t to time $t + dt$ generated through standard marking-to-market mechanisms as a function of the dynamic trading strategy in futures contracts. This feature of the futures

⁹ - The Black formula is originally established for the pricing of options on forward contracts. It can also be applied to the pricing of options on futures contracts if the nominal short-term rate is constant because in this context, forward prices are equal to futures prices.

¹⁰ - We show below that for CARA preferences the optimal hedging strategy and corresponding option price do not depend on V_0 .

contract makes its value equal zero at each date.¹¹ On the other hand, the term $rV_t dt$ accounts for the investment in the risk-free asset, which is the only cash instrument in the model.

In this incomplete market setting, the self-financing strategy leads in general to a terminal surplus (or deficit) denoted by $V_T - (X_T - K)^+$ at time T . We assume that the investor has preferences over the net final payoff, *i.e.*, final wealth net of option payoff, where the preferences are captured by a standard constant absolute risk aversion (CARA) function $u(x) = -e^{-\gamma x}$.¹² The utility maximization program can thus be written as:

$$\max_{(n_t)_{t \geq 0}} \mathbb{E}^{\mathbb{P}} \left[-e^{-\gamma(V_T - (X_T - K)^+)} \right] \quad (9)$$

for some absolute risk aversion parameter γ . Let \tilde{n} denote the solution to this program. Similarly, let \tilde{n}^0 denote the optimal number of futures contracts when no option is written, that is, when the program reads:

$$\max_{(n_t)_{t \geq 0}} \mathbb{E}^{\mathbb{P}} \left[-e^{-\gamma V_T} \right]$$

We define the *indifference hedging strategy* n^* as the difference in the optimal allocation policy for an agent holding a short position in the option contract versus the allocation policy for an agent who holds no such position:

$$n^* = \tilde{n} - \tilde{n}^0$$

The following proposition provides the hedging strategy and the option price for a given absolute risk aversion level γ .

Proposition 1 *The utility-maximizing strategy \tilde{n} and the indifference hedging strategy n^* are respectively given by:*

$$\tilde{n}_t = \frac{\mu_F}{\sigma_F^2 F_t} \frac{e^{-r(T-t)}}{\gamma} + \frac{H_X \sigma_X X_t}{\sigma_F F_t} \rho_{FX} + H_F \quad (10)$$

$$n_t^* = \frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} b_X(t, F_t, X_t) + b_F(t, F_t, X_t) \quad (11)$$

where $b(t, F_t, X_t)$ denotes the indifference price of the option, which is independent of current wealth, and is given by:

$$b(t, F_t, X_t) = H(t, F_t, X_t) - H^0(t, F_t, X_t)$$

In these expressions H and H^0 are solutions to the same partial differential equation (p.d.e.) with different terminal conditions given in appendix A (see (A.5)).

Proof. See appendix A.

Several comments are in order. As expected, we find that the optimal allocation decisions depend on two state variables: the spot price X for the underlying asset, and the futures price F . The utility-maximizing strategy (10) first involves the familiar speculative demand for futures contract $\frac{\mu_F}{\sigma_F^2 F_t} \frac{e^{-r(T-t)}}{\gamma}$, a term that vanishes for an infinite risk aversion. It also involves two hedging demands, $\frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} H_X$ and H_F . The second term $\frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} H_X$ in the optimal strategy (10) is a generalization of the standard Black-Scholes delta-neutral strategy, which is extended to account for the presence of imperfect correlation between the underlying asset and the substitute, with a correction factor given by $\frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX}$. If the correlation between changes in the underlying asset price and the futures price happens to be zero, then this hedging demand disappears, as it should. On the other hand, the third term H_F in the right-hand side of equation (10) is entirely new, and can

11 - In practice, marking-to-market is performed on a daily, as opposed to continuous, basis. The assumption of continuous margin calls is for computational tractability.
12 - See Duffie et al. (1997) for general results on hedging strategies with unspanned endowment risk and more general HARA preferences.

formally be interpreted as an intertemporal hedging demand against basis risk, that is, against unexpected changes in the difference between futures and spot prices. The indifference hedging strategy involves the same hedging demands, but the partial derivatives (deltas) of the function H are replaced by those of the indifference price b . Moreover, the speculative demand for futures contracts is eliminated. Note also that we recover the Black formula in case $T_0 = T$, so that the option price only depends on F_t and the delta b_X is identically zero. The Black formula can also be recovered as a specific case for $\sigma_D = 0$ (see appendix A).

We now show that analytical expressions can be obtained in the limit of a vanishing risk aversion ($\gamma \rightarrow 0$). Such analytical expressions are of substantial interest from the practical perspective because they allow for a fast and accurate computation of the option price and hedging strategy. This justifies the focus on the special case $\gamma \rightarrow 0$ in the empirical application in section 2.¹³ Of course, the option price and hedging strategy can also be obtained via standard numerical methods for a positive γ . The following proposition provides the analytical expression for the price and for the indifference hedging strategy.

Proposition 2 *Assume that the risk aversion shrinks to zero. Then the limit of the indifference price, denoted by $c(t, F_t, X_t)$, and the corresponding hedging strategy, denoted by \hat{n}_t , are respectively given by:*

$$c(t, F_t, X_t) = \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T-t)} (X_T - K)^+]$$

$$\hat{n}_t = c_F(t, F_t, X_t) + \frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} c_X(t, F_t, X_t)$$

Here \mathbb{Q} is the martingale measure defined by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left[- \int_0^T \frac{\mu_F}{\sigma_F} dz_t^F - \int_0^T \frac{\mu_F^2}{2\sigma_F^2} dt \right]$$

In closed form, we have:

$$c(t, F_t, X_t) = e^{\mu_{t,T} + \frac{1}{2}\Sigma_{t,T} - r(T-t)} \mathcal{N}(d_{1,t}) - Ke^{-r(T-t)} \mathcal{N}(d_{2,t})$$

$$\hat{n}_t = \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha + \frac{\rho_{FX}\sigma_X}{\sigma_F} \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha \right] \frac{e^{\mu_{t,T} + \frac{1}{2}\Sigma_{t,T} - r(T-t)}}{F_t} \mathcal{N}(d_{1,t}) \quad (12)$$

where:

$$d_{1,t} = \frac{1}{\sqrt{\Sigma_{t,T}}} [\mu_{t,T} + \Sigma_{t,T} - \ln K], \quad d_{2,t} = d_{1,t} - \sqrt{\Sigma_{t,T}}$$

and $\mu_{t,T}$ and $\Sigma_{t,T}$ are the first two moments of $\ln X_T$ given \mathcal{F}_t :

$$\mu_{t,T} = \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha \right] \ln F_t + \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha \ln X_t + (T - t)b_1$$

$$+ \frac{1}{1 - \alpha} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^\alpha (T_0 - t) - (T_0 - T) \right] (b_2 - b_1)$$

$$\Sigma_{t,T} = (T - t)\sigma_F^2 + \frac{2\sigma_F(\rho_{FX}\sigma_X - \sigma_F)}{1 - \alpha} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^\alpha (T_0 - t) - (T_0 - T) \right]$$

$$+ \frac{\sigma_F^2 - 2\rho_{FX}\sigma_F\sigma_X + \sigma_X^2}{1 - 2\alpha} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^{2\alpha} (T_0 - t) - (T_0 - T) \right]$$

¹³ - Another possible justification for the focus on vanishing risk-aversion is the finding that the optimal hedging strategy presented in proposition 2 is also a "locally risk-minimizing" strategy in the sense of Föllmer and Schweizer (1990). Intuitively, local risk-minimization is a heuristic criterion focusing on minimizing at each date the expected squared profit and loss generated by the replicating portfolio strategy over the next infinitesimal period. More details and a formal proof of this result can be obtained from the authors upon request.

with

$$b_1 = -\frac{\sigma_F^2}{2}, \quad b_2 = \mu_X - \frac{\sigma_X^2}{2} - \frac{\rho_{FX}\sigma_X}{\sigma_F} \left[\mu_X + \frac{1}{2} (\sigma_D^2 + 2\rho\sigma_X\sigma_D) \right], \quad \alpha = \frac{\rho_{FX}\sigma_X a}{\sigma_F}$$

Proof. See appendix B.

The expression for the indifference price has the same functional form as the Black formula, but the coefficients have a more complex expression, because (i) basis risk is not entirely spanned, and (ii) the option and the futures contracts have different maturities. In the case where $T_o = T$, the indifference price coincides with the Black price (6). In the case where there is no basis risk (i.e., when $\sigma_D = 0$), it is also equal to the unique no-arbitrage price, which is still given by a Black-like option pricing formula, but with slightly more complicated coefficients compared to the standard expression in (6), due to the correction for the difference in maturities. Note that the option price and corresponding hedging strategy depend on the underlying asset expected return μ_X , a question we revisit in section 2 when performing an empirical calibration of the model.

Having obtained an analytical expression for the indifference hedging strategy, one may wonder how to assess the accuracy of the replication. One natural, albeit heuristic, criterion is the quadratic replication error:¹⁴

$$R_0 = \mathbb{E} \left[e^{-2rT} (V_T - (X_T - K)^+)^2 \right] \quad (13)$$

The following proposition provides an expression for the quadratic error when the indifference hedging strategy is implemented.

Proposition 3 *The quadratic error associated with the hedging strategy of proposition 2 can be expressed as:*

$$R_0 = [V_0 - c(0, F_0, X_0)]^2 + \sigma_X^2 (1 - \rho_{FX}^2) \mathbb{E} \left[\int_0^T e^{-2rt} c_X(0, F_t, X_t)^2 dt \right]$$

Proof. See appendix C.

If the initial wealth V_0 is equal to the reservation price $c(0, F_0, X_0)$ for the option, then replication error is fully explained by the second term on the right hand side of the expression for R_0 . This term vanishes if the underlying asset of the option is perfectly correlated with the futures price ($\rho_{FX}^2 = 1$), i.e., when there is no basis risk. It is also increasing in the delta of the option: the intuition is that large deltas imply large positions in the futures contracts that magnify the effect of imperfect correlation. It also depends on the parameters σ_D , a and ρ , and also on the respective maturities of the option and the futures contract. In the next section, we provide a numerical analysis of the impact of changes in those parameters on the replication error measured by the squared-root $\sqrt{R_0}$ of the quadratic error.

2. Empirical Analysis

This section presents an empirical application of the model of sub-section 1.2. More specifically, we consider the case of a three-month option written on the S&P 500 index. This option can be hedged by trading in futures contracts, with trading costs substantially lower than those of trading in some passive mutual fund of exchange-traded fund replicating the S&P 500 index. A natural choice, if and when feasible, would be to take a contract of maturity equal to three months as this would make the problem of replicating the option trivial, as noted in section 1. In what follows, we consider more general situations where the option is hedged with a futures contract maturing after the option. Even if contracts with such a maturity are not readily available on the market, they can be synthesized by rolling over contracts of shorter maturities. In the end, we consider the case where the maturity date of the last contract involved in the roll-

¹⁴ - See Schweizer (2001) for a discussion of heuristic option replication techniques based on the minimization of R_0 . One shortcoming of this criterion is that it penalizes positive net wealth and negative net wealth equally.

over exceeds the expiration date of the option by one, three or six months. Contracts with longer maturities would probably not be liquid enough to provide good substitutes.

2.1 Calibration of the Model

We first calibrate model parameters to S&P 500 index and index futures return series. S&P 500 index daily returns are obtained from Datastream. The Commodity Research Bureau provides data for futures contracts written on the S&P 500, with complete data for two-year futures contracts over the period 2000–2009.¹⁵ Every year, four different maturities are available, which correspond respectively to March, June, September and December. In the end, we consider ten contracts, all maturing in March, from March 2000 to March 2009, for which a two-year daily time series is available. This gives us ten rolling windows with one year overlap. Figure 1 displays the index and the futures price series for three of these windows. These three examples are meant to illustrate three different rates of convergence for the basis, *i.e.*, the log-difference between the futures and the spot prices, to zero. The convergence was relatively slow over the period 1999–2001, but very fast over the period 2003–2005, with the two time-series very close to each other already early on in the futures contract lifetime. Finally, the most recent window, 2007–2009, is an example of an intermediate speed of convergence.

Given the dynamics (1) and (3), the vector $(\ln F_t, \ln X_t)$ is Gaussian under the physical probability measure \mathbb{P} , which enables us to obtain an analytical expression for the likelihood of observations in terms of the continuous-time parameters $\mu_X, \sigma_X, \sigma_F, \rho_{FX}$ and a . The parameters for the basis process, σ_D, ρ_{SD} and ρ_{DX} , can then be recovered using equations (4) and (5). Appendix D provides more details as well as the expression for the likelihood function. Maximizing this function with respect to the five parameters is numerically feasible, but it leads to unreasonable values, ranging from -22% to 14% , for the drift μ_X . In fact, expected return estimates are well known to be noisy (see Merton (1980)), and the maximum likelihood estimator of the drift is highly sensitive to the chosen sample. To address this problem, we have tested a restricted form of the model, where μ_X is set to some arbitrary value, namely, 10% . We have also taken the expected return restricted value to be equal to the risk free rate (3% in our base case) and found parameter estimates to be very robust with respect to this change (see table 1).¹⁶ The results we obtain are reported in table 1. As could be expected, the estimated value for the speed on mean-reversion a turns out to be higher when the convergence of the basis process to zero is fast (see in particular the column related to the sample period 2003–2005 in table 1).

Having estimated the parameters over each sample period, we construct a base-case set of parameter values by taking the median value for each parameter estimate across all sample periods. The median value is preferred to the mean value because it is more robust to the presence of a single extreme value. For the parameter a , this property turns out to be important since the estimate over the period 2003–2005 appears to be an outlier compared to what is obtained for other sample periods. This base-case set of parameter values is summarized in table 2. All subsequent results have been obtained by generating 20,000 scenarios for the processes X and D according to the dynamics of section 1 and by computing R_0 from its definition (see equation (13)).

2.2 Numerical Results

We first analyze the quadratic error R_0 induced by the optimal hedging strategy of proposition 2. The initial wealth V_0 is taken equal to $c(0, F_0, X_0)$.¹⁷ For aforementioned reasons, we consider futures contracts with a maturity date T_0 strictly greater than the option contract maturity date T , with the distance $T_0 - T$ taken to be one month, three months or six months, respectively. Intuitively, we expect that increasing the difference between the futures contract maturity date T_0 and the option contract maturity date T should result in a higher replication error, everything

¹⁵ - In an attempt to use the longest possible sample, we choose to use two-year futures contract to calibrate the basis risk model, even if we use futures contracts with shorter maturity in the analysis of the performance of the option hedging strategy.

¹⁶ - Empirical evidence suggests that the parameter estimates depend on the sample period, and we leave for further research the solution to the hedging and pricing problem in the presence of such stochastically time-varying parameters. The presence of stochastic parameters is expected to induce additional hedging demands in the indifference hedging strategy of proposition 4.

else equal. Similarly, we expect that a decrease in the speed of mean-reversion parameter a , or an increase in basis risk σ_D volatility should lead to an increase in the replication error.

We confirm these intuitions both when the replication error is expressed in absolute terms (see table 3) and when it is expressed in relative terms with respect to the initial reservation price (see table 4). Focusing on reasonable parameter values, for example $a = 3$ and $\sigma_D = 2.5\%$ (values chosen to be close to the estimated values (see table 2), the relative replication error amounts to 6.34% when the four-month contract is used, 9.34% with the six-month contract and 12.78% with the one-year contract. When there is no basis risk ($\sigma_D = 0$), the replication error is zero since the option can be perfectly replicated using the Black strategy (7), or some simple variant of the Black strategy accounting for the difference in maturities between the option contract and the futures contract.¹⁸ In the presence of basis risk, on the other hand, using the Black strategy, which appears to be the standard hedging strategy used in practice, generates a strictly positive replication error. Table 5 compares the error that results from following the Black strategy to the residual error generated by the optimal strategy. The increase in error is a decreasing function of a , which means that when the basis slowly converges to zero, using the Black strategy is even more costly than when the convergence is fast. On the other hand, the cost of using the heuristic strategy – still measured by the increase in error – is a decreasing function of σ_D , because the residual replication error is a fast-growing function of σ_D (see table 3). The benefits from using the optimal strategy as opposed to the Black strategy may become exceedingly high (with an increase in error of more than 100%), in particular for the lowest values of σ_D , in which case the replication error for the optimal strategy is close to zero. For parameter values close to the base case, we find that using the Black strategy would still lead to a significant increase in error of 27.75% for $a = 3$ and $\sigma_D = 2.5\%$ and $T_0 - T = 3$ months).

Although it performs significantly better than the Black strategy when parameter values are perfectly observed, the indifference hedging strategy presented in proposition 2 suffers from a major drawback from a practical implementation perspective: it requires the use of the drift parameter μ_X for the underlying asset. Given the aforementioned difficulty in estimating expected return parameters, one might wonder what would be the impact of an estimation error on the performance of the optimal strategy. In particular, if the estimation error led to a strong deterioration of the optimal hedging strategy, using the Black strategy might turn out to be the preferred option after all. To analyze this question, we now implement the optimal strategy while taking the expected return parameter equal to the risk-free rate, while stochastic scenarios for the process X are still generated with the actual drift taken at the base case value of 10%. In other words, the writer of the option still follows the strategy of proposition 2 but incorrectly replaces the drift μ_X with the risk-free rate r wherever needed.

Table 6 presents the percentage increase in the replication error $\sqrt{R_0}$ with respect to the case where the true value is used. It appears that even a strong underestimation of the drift leads to a very small increase in the replication error, an increase found to be less than 1% for all parameter values we consider. In view of the numbers in table 5, the performance of the optimal hedging strategy therefore appears to be robust with respect to a measurement error in the drift of the underlying asset, and it strongly dominates the heuristic strategy based on the Black formula, even if one accounts for the presence of estimation errors in expected return parameter estimates.

3. Introducing Cross-Hedge Risk

In this section we consider an extension of the model of section 1 in which we relax the restrictive assumption that the underlying asset for the futures contract is also the underlying asset for the option contract. In other words, we recognize that, in equity derivatives trading, it is often the

18 - We implement the hedging strategies with discrete trading so that the final P&L may be not exactly zero in principle. Numerically, we have found only minuscule errors, which have been rounded to zero.

case that the option may be written on some not easily tradable asset X (e.g., the S&P 100 index or some other basket of stocks) while the traded futures contract is written on some (also not easily traded) imperfectly correlated substitute asset S (e.g., the S&P 500 index). There are therefore two sources of incompleteness in this general situation, namely, (i) cross-hedge risk, arising from the use of two different underlying assets for the option and for the futures contract, and (ii) basis risk, arising from the imperfect correlation between changes in the spot price S and changes in the futures price F .

3.1 Economic Framework

The substitute asset price S is assumed to follow a geometric Brownian motion:

$$dS_t = S_t [\mu_S dt + \sigma_S dz_t^S] \quad (14)$$

and the log difference $D = \ln(F/S)$ between the futures and the spot prices of S is again assumed to evolve as a Brownian bridge process:

$$dD_t = \frac{-aD_t}{T_0 - t} dt + \sigma_D dz_t^D \quad (15)$$

The dynamic for X is still given by (1), and the pairwise correlations between the Brownian motions z^S , z^D and z^X are denoted by ρ_{SD} , ρ_{SX} and ρ_{DX} . In this section, the information set to which the investor has access at time t is still denoted by \mathcal{F}_t and is the augmented version of the sigma-algebra generated by the processes z^S , z^X and z^D . In other words, we assume that the investor observes futures prices as well as the prices for the underlying asset and the substitute.

In this model, the volatility of the futures return and the instantaneous correlations of the futures returns with the return on S and X are respectively given by:

$$\sigma_F = \sqrt{\sigma_S^2 + \sigma_D^2 + 2\rho_{SD}\sigma_S\sigma_D} \quad (16)$$

$$\rho_{FS} = \frac{\sigma_S}{\sigma_F} + \frac{\sigma_D\rho_{SD}}{\sigma_F} \quad (17)$$

$$\rho_{FX} = \frac{\sigma_D}{\sigma_F}\rho_{DX} + \frac{\sigma_S}{\sigma_F}\rho_{SX} \quad (18)$$

In the absence of cross-hedge risk (i.e., if $\rho_{SX} = 1$), we have that:

$$X_T = f(T)S_T^{\frac{\sigma_X}{\sigma_S}}$$

where $f(T) = X_0 S_0^{-\frac{\sigma_X}{\sigma_S}} e^{\left[\mu_X - \frac{\sigma_X^2}{2} - \frac{\sigma_X}{\sigma_S} \left(\mu_S - \frac{\sigma_S^2}{2}\right)\right]T}$. Hence the call option on X can be regarded as a polynomial option written on S , the underlying asset for the futures contract. This allows us to revert to the framework of section 1, where the presence of basis risk is the only source of replication error. In general, however, the presence of unspanned basis risk and the unavailability of a perfect substitute for the underlying asset imply the existence of some additional replication error.

3.2 Option Replication with Basis Risk and Cross-Hedge Risk

As in the case with no cross-hedge risk, we consider the case of an option writer attempting to maximize expected utility from terminal wealth net of the option payoff (see program (9)). The optimal hedging strategy and corresponding price are given in the next proposition.

Proposition 4 Let $b(t, F_t, X_t, S_t)$ be the indifference price of the option. Then the indifference hedging strategy is given by:

$$n_t^* = b_F(t, F_t, X_t, S_t) + \frac{\sigma_S S_t}{\sigma_F F_t} \rho_{FS} b_S(t, F_t, X_t, S_t) + \frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} b_X(t, F_t, X_t, S_t)$$

and b can be obtained as the solution to a p.d.e. with terminal condition $b(T, F_T, X_T, S_T) = (X_T - K)^+$.

Proof. The proof of this result is strictly similar to that of proposition 1 with one additional variable, which is the price of the underlying S . Further details, as well as the p.d.e. satisfied by b , can be obtained from the authors upon request.

In the presence of cross-hedge risk, the indifference hedging strategy appears to be given by the sum of three terms. The first two terms are the same as in proposition 1, where the optimal hedging strategy was obtained in the absence of cross-hedge risk. The last term precisely arises because of the imperfect correlation between the two underlying assets.¹⁹ Since there are three state variables driving the indifference price, the derivation of the optimal hedging strategy requires the estimation of three delta terms at each point in time, which increases the computational burden if one uses numerical procedures. The following proposition shows that fully analytical expressions can be recovered even in this more general setting when γ converges to zero.

Proposition 5 Let γ go to zero. Then the indifference price and the indifference hedging strategy converge respectively to:

$$c(t, F_t, X_t, S_t) = e^{\mu_{t,T} - r(T-t) + \frac{1}{2} \Sigma_{t,T}} \mathcal{N}(d_{1,t}) - K e^{-r(T-t)} \mathcal{N}(d_{2,t})$$

$$\hat{n}_t = \frac{e^{\mu_{t,T} + \frac{1}{2} \Sigma_{t,T} - r(T-t)}}{F_t} \mathcal{N}(d_{1,t}) \left[\left(1 - \frac{\rho_{FS} \sigma_S}{\sigma_F} \right) \frac{\alpha_2}{\alpha_1} \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} \right] + \frac{\sigma_X \rho_{FX}}{\sigma_F} \right]$$

with $d_{1,t} = \frac{\mu_{t,T} + \Sigma_{t,T} - \ln K}{\sqrt{\Sigma_{t,T}}}$ and $d_{2,t} = \frac{\mu_{t,T} - \ln K}{\sqrt{\Sigma_{t,T}}}$.

The first derivatives of the option price with respect to each of the three state variables are given by:

$$\begin{aligned} c_X(t, F_t, X_t) &= \frac{e^{\mu_{t,T} + \frac{1}{2} \Sigma_{t,T} - r(T-t)}}{X_t} \mathcal{N}(d_{1,t}) \\ c_F(t, F_t, X_t) &= \frac{\alpha_2}{\alpha_1} \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} \right] \frac{e^{\mu_{t,T} + \frac{1}{2} \Sigma_{t,T} - r(T-t)}}{F_t} \mathcal{N}(d_{1,t}) \\ c_S(t, F_t, X_t) &= -\frac{\alpha_2}{\alpha_1} \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} \right] \frac{e^{\mu_{t,T} + \frac{1}{2} \Sigma_{t,T} - r(T-t)}}{S_t} \mathcal{N}(d_{1,t}) \end{aligned}$$

while the first two conditional moments of $\ln X_T$ given \mathcal{F}_t , $\mu_{t,T}$ and $\Sigma_{t,T}$, are given by:

$$\begin{aligned} \mu_{t,T} &= \frac{\alpha_2}{\alpha_1} \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} \right] \ln \frac{F_t}{S_t} + \ln X_t + b_3(T-t) \\ &\quad + \frac{\alpha_2}{\alpha_1} (b_1 - b_2) \left[T - t - \frac{1}{1 - \alpha_1} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} (T_0 - t) - (T_0 - T) \right] \right] \end{aligned} \quad (19)$$

and:

$$\begin{aligned} \Sigma_{t,T} &= \left[\sigma_X^2 + \left(\frac{\alpha_2}{\alpha_1} \right)^2 (\sigma_F^2 + \sigma_S^2 - 2\sigma_F \sigma_S \rho_{FS}) + 2 \frac{\alpha_2}{\alpha_1} (\sigma_F \sigma_X \rho_{FX} - \sigma_S \sigma_X \rho_{SX}) \right] (T-t) \\ &\quad + \left(\frac{\alpha_2}{\alpha_1} \right)^2 \frac{\sigma_F^2 + \sigma_S^2 - 2\sigma_F \sigma_S \rho_{FS}}{1 - 2\alpha_1} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^{2\alpha_1} (T_0 - t) - (T_0 - T) \right] \\ &\quad - 2 \frac{\alpha_2}{\alpha_1} \left[\frac{\alpha_2}{\alpha_1} (\sigma_F^2 + \sigma_S^2 - 2\sigma_S \sigma_F \rho_{FS}) + \sigma_F \sigma_X \rho_{FX} - \sigma_S \sigma_X \rho_{SX} \right] \\ &\quad \times \frac{1}{1 - \alpha_1} \left[\left(\frac{T_0 - T}{T_0 - t} \right)^{\alpha_1} (T_0 - t) - (T_0 - T) \right] \end{aligned} \quad (20)$$

where:

$$\alpha_1 = \frac{\rho_{FS}\sigma_S}{\sigma_F}a, \quad \alpha_2 = \frac{\rho_{FX}\sigma_X}{\sigma_F}a$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -\frac{\sigma_F^2}{2} \\ \mu_S - \frac{\sigma_S^2}{2} - \frac{\sigma_S\rho_{FS}}{\sigma_F} \left[\mu_S + \frac{1}{2}(\sigma_D^2 + 2\rho_{SD}\sigma_S\sigma_D) \right] \\ \mu_X - \frac{\sigma_X^2}{2} - \frac{\sigma_X\rho_{FX}}{\sigma_F} \left[\mu_S + \frac{1}{2}(\sigma_D^2 + 2\rho_{SD}\sigma_S\sigma_D) \right] \end{pmatrix}$$

Proof. The proof of this proposition is similar to that of propositions 1 and 2, up to the presence of the third state variable S .

The reservation price is formally similar to the indifference price in the absence of cross-hedge risk, with the only difference being the adjustment for the imperfect correlation between the two underlying assets, and for the differences in their drift and volatility parameters. Again, having analytical expressions for the reservation price and the indifference hedging strategy greatly simplifies the practical implementation of the strategy, since no numerical scheme such as finite differences or Monte-Carlo simulation is needed.

4. Conclusion

In this paper, we analyze the problem of option pricing and hedging when a futures contract written on the underlying asset, or on some imperfect substitute for the underlying asset, is used in the dynamic replication of the option payoff. Using a Brownian bridge process to model basis risk, we obtain expressions for the dynamic hedging strategy and the corresponding price consistent with an expected utility maximization criterion. For reasonable parameter values, we find the replication error implied by the optimal strategy to be substantially lower than that implied by the Black pricing and hedging rule. These results confirm that the use of the optimal strategy is likely to generate economically meaningful welfare gains.

A number of related dimensions could be further explored. In particular, one would like to assess the impact of the introduction of time-varying interest rates, volatility or correlation parameters. Introducing interest rate uncertainty would be of prime relevance for the purpose of analyzing optimal hedging strategies for fixed-income derivatives based on trading in interest rate futures. Additionally, it would be useful to analyze the impact of the introduction of additional forms of incompleteness, such as the presence of jumps in asset prices. We leave these problems for further research.

Appendix A: Proof of Proposition 1

The indirect utility function for the program (9) is defined as:

$$J(t, V_t, F_t, X_t) = \max_{(n_s)_{s \geq t}} \mathbb{E}_t \left[-e^{-\gamma(V_T - (X_T - K)^+)} \right]$$

In the presence of basis risk, the market is incomplete. Hence there exist infinitely many martingale measures consistent with the absence of arbitrage opportunities, with conditional densities with respect to \mathbb{P} given by:

$$Q_t = \exp \left[- \int_0^t \left(\frac{\mu_F}{\sigma_F} dz_s^F + \chi_s dW_s \right) - \frac{1}{2} \int_0^t \left(\frac{\mu_F^2}{\sigma_F^2} + \chi_s^2 \right) ds \right], \quad 0 \leq t \leq T$$

where $dW_t = \frac{1}{\sqrt{1-\rho_{FX}^2}} dz_t^X - \frac{\rho_{FX}}{\sqrt{1-\rho_{FX}^2}} dz_t^F$ is a \mathbb{P} -Brownian motion.

χ denotes the market price of risk associated with W . In what follows, we let $Q_{t,T}$ denote the ratio Q_T/Q_t . For each density, one can formulate a static maximization problem at time t , with the control variable being the terminal contingent wealth V_T :

$$\max_{V_T} \mathbb{E}_t \left[-e^{-\gamma(V_T - (X_T - K)^+)} \right], \quad \text{s.t.} \quad \mathbb{E}_t \left[e^{-r(T-t)} Q_{t,T} V_T \right] = V_t \quad (\text{A.1})$$

Any optimal wealth in (A.1) is a candidate optimal terminal wealth in (9), but it may not always be attainable. As shown by He and Pearson (1991), the unique optimal attainable wealth is obtained by minimizing the value of the static problem, subject to the budget constraint. First-order optimality conditions in (A.1) show that the value function for the static problem is given by:

$$J^{(S)}(t, V_t, F_t, X_t) = \inf_{\chi, \eta} \left\{ -\mathbb{E}_t \left[\frac{\eta Q_{t,T} e^{-r(T-t)}}{\gamma} \right] \right\} \quad (\text{A.2})$$

where the scalar η can be eliminated from (A.2) using the budget constraint. This leads to the following minimum value for the static problem, which is also the value function for the original dynamic problem:

$$J(t, V_t, F_t, X_t) = -e^{-\gamma e^{r(T-t)} [V_t - H(t, F_t, X_t)]} \quad (\text{A.3})$$

$$H(t, F_t, X_t) = \max_{\chi} e^{-r(T-t)} \mathbb{E}_t \left[Q_{t,T} \left((X_T - K)^+ - \frac{\ln Q_{t,T}}{\gamma} \right) \right] \quad (\text{A.4})$$

Equation (A.3) provides a separation result for the utility function. We first use it to derive an expression for the indifference price. *A priori*, this price depends on (t, V_t, F_t, X_t) , and is implicitly defined by:

$$J(t, V_t, F_t, X_t) = J^0(t, V_t - b(t, V_t, F_t, X_t), F_t, X_t)$$

where J^0 is the indirect utility function when no option is written. J^0 can be obtained by replacing the payoff $(X_T - K)^+$ with 0 in (A.4). We thus obtain that:

$$b(t, V_t, F_t, X_t) = H(t, F_t, X_t) - H^0(t, F_t, X_t)$$

where:

$$H^0(t, F_t, X_t) = -\frac{1}{\gamma} \min_{\chi} e^{-r(T-t)} \mathbb{E}_t [Q_{t,T} \ln Q_{t,T}]$$

In particular, the indifference price proves to be independent of current wealth V_t . With a slight abuse of notation, we now denote it by $b(t, F_t, X_t)$.

Consider now the Hamilton-Jacobi-Bellman equation for the value function J . The first-order optimality condition with respect to n_t yields the expression for the optimal number of futures contracts:

$$\tilde{n}_t = H_F + \frac{H_X \sigma_X X_t}{\sigma_F F_t} \rho_{FX} + \frac{\mu_F}{\sigma_F^2 F_t} \frac{e^{-r(T-t)}}{\gamma}$$

The optimal number of futures contract to be held in the absence of the option admits a similar expression, where H is replaced by H^0 . Hence the expression (11).

Plugging the expression for \tilde{n} back into the HJB equation, one obtains the p.d.e. satisfied by J . Substituting (A.3) into this p.d.e. then yields a p.d.e. satisfied by H :

$$0 = H_t - rH + H_X X \left[\mu_X - \frac{\sigma_X}{\sigma_F} \rho_{FX} \mu_F \right] + \frac{1}{2} \gamma e^{r(T-t)} \sigma_X^2 H_X^2 (1 - \rho_{FX}^2) + \frac{1}{2} H_{XX} \sigma_X^2 X^2 + \frac{1}{2} H_{FF} \sigma_F^2 F^2 + H_{FX} \sigma_X X \sigma_F F \rho_{FX} - \frac{\mu_F^2}{2\sigma_F^2} \frac{e^{-r(T-t)}}{\gamma} \quad (\text{A.5})$$

with the terminal condition $H(T, X_T, F_T) = (X_T - K)^+$. H^0 is a solution to the same p.d.e., with the terminal condition $H^0(T, F_T, X_T) = 0$. Hence b solves:

$$0 = b_t - rb + b_X X \left(\mu_X - \frac{\sigma_X}{\sigma_F} \rho_{FX} \mu_F \right) + \frac{1}{2} \gamma e^{r(T-t)} (1 - \rho_{FX}^2) \sigma_X^2 X^2 \left[H_X^2 - (H_X^0)^2 \right] + \frac{1}{2} b_{XX} \sigma_X^2 X^2 + \frac{1}{2} b_{FF} \sigma_F^2 F^2 + b_{FX} \sigma_X X \sigma_F F \rho_{FX} \quad (\text{A.6})$$

with the terminal condition $b(T, F_T, X_T) = (X_T - K)^+$.

When $\sigma_D = 0$, the basis process D is deterministic, so that F is a function of time only. Equation (A.5) with the terminal condition $H(T, F_T, X_T) = (F_T e^{-DT} - K)^+$ then implies that $H_X = H_X^0 = 0$, so that equation (A.6) reduces to:

$$0 = b_t - rb + \frac{1}{2} \sigma_F^2 F^2 b_{FF}$$

with the terminal condition $b(T, F_T, X_T) = e^{-DT} (F_T - K e^{DT})^+$. The solution to this equation is e^{-DT} times the Black price of a call option written on F with strike price $K e^{DT}$.

Appendix B: Proof of Proposition 2

If $\gamma = 0$, then equation (A.6) becomes:

$$0 = b_t - rb + b_X X \left(\mu_X - \frac{\sigma_X}{\sigma_F} \rho_{FX} \mu_F \right) + \frac{1}{2} b_{XX} \sigma_X^2 X^2 + \frac{1}{2} b_{FF} \sigma_F^2 F^2 + b_{FX} \sigma_X X \sigma_F F \rho_{FX}$$

It follows from Feynman Kac's theorem (see Karatzas and Shreve (1991)) that the solution c to this equation admits a probabilistic representation:

$$c(t, F_t, X_t) = \mathbb{E}_t^{\mathbb{Q}} [e^{-r(T-t)} (X_T - K)^+]$$

We now compute c in closed form. Let us define a state vector \mathbf{Z} as $\mathbf{Z}_t = \left(\ln F_t \quad \ln X_t \right)'$.

The dynamics of F and X under \mathbb{Q} can be written in a synthetic form as:

$$d\mathbf{Z}_t = (\mathbf{B} + \mathbf{A}_t \mathbf{Z}_t) dt + \boldsymbol{\sigma}' dz_t^{\mathbb{Q}} \quad (\text{B.1})$$

where we have set:

$$B = \begin{pmatrix} -\frac{\sigma_F^2}{2} \\ \mu_X - \frac{\sigma_X^2}{2} - \frac{\rho_{FX}\sigma_X}{\sigma_F} \left[\mu_X + \frac{1}{2} (\sigma_D^2 + 2\rho\sigma_D\sigma_X) \right] \end{pmatrix}, \quad A_t = \begin{pmatrix} 0 & 0 \\ \frac{\rho_{FX}\sigma_X}{\sigma_F} \frac{a}{T_0-t} & -\frac{\rho_{FX}\sigma_X}{\sigma_F} \frac{a}{T_0-t} \end{pmatrix}$$

$z^{\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion and σ is a matrix such that:

$$\sigma' \sigma = \begin{pmatrix} \sigma_F^2 & \rho_{FX}\sigma_X\sigma_F \\ \rho_{FX}\sigma_X\sigma_F & \sigma_X^2 \end{pmatrix}$$

Integrating the dynamics (B.1) from t to T , we obtain that $\ln X_T$ is conditionally normally distributed, with mean and variance given by:

$$\mu_{t,T} = (\mathbb{E}_t[\mathbf{Z}_T])_2, \quad \Sigma_{t,T} = (\mathbb{V}_t[\mathbf{Z}_T])_{2,2}$$

The log-normality of $\ln X_T$ enables us to obtain the following analytical expression for the reservation price:

$$c(t, F_t, X_t) = e^{\mu_{t,T} + \frac{1}{2}\Sigma_{t,T} - r(T-t)} \mathcal{N}(d_{1,t}) - K e^{-r(T-t)} \mathcal{N}(d_{2,t})$$

where $d_{1,t}$ and $d_{2,t}$ are given in the proposition. The limit of the indifference hedging strategy as γ goes to zero is given by:

$$n_t^* = c_F(t, F_t, X_t) + \frac{\sigma_X X_t}{\sigma_F F_t} \rho_{FX} c_X(t, F_t, X_t) \quad (\text{B.2})$$

The first-order partial derivatives of the option price with respect to the underlying and the futures prices are easily obtained as:

$$c_X = \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha \frac{e^{\mu_{t,T} + \frac{1}{2}\Sigma_{t,T} - r(T-t)}}{X_t} \mathcal{N}(d_{1,t})$$

$$c_F = \left[1 - \left(\frac{T_0 - T}{T_0 - t} \right)^\alpha \right] \frac{e^{\mu_{t,T} + \frac{1}{2}\Sigma_{t,T} - r(T-t)}}{F_t} \mathcal{N}(d_{1,t})$$

Plugging these expressions back into (B.2) we obtain the expression for \hat{n}_t .

Appendix C: Proof of Proposition 3

We now compute the quadratic error associated with the hedging strategy. We start with the dynamics of F and X under \mathbb{Q} :

$$\frac{dF_t}{F_t} = \sigma_F dz_t^{F,\mathbb{Q}}$$

$$\frac{dX_t}{X_t} = \left[\mu_X - \sigma_X \rho_{FX} \frac{\mu_F}{\sigma_F} \right] dt + \sigma_X dz_t^{X,\mathbb{Q}}$$

where $z^{X,\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion defined $dz_t^{X,\mathbb{Q}} = dz_t^X + \frac{\mu_F}{\sigma_F} \rho_{FX} dt$.

The quantity $e^{-rt} c(t, F_t, X_t)$ follows a martingale under \mathbb{Q} , so that, for any $t \geq T$:

$$e^{-rt} c(t, F_t, X_t) = c(0, F_0, X_0) + \int_0^t e^{-rs} [c_F \sigma_F F_s dz_s^{F,\mathbb{Q}} + c_X \sigma_X X_s dz_s^{X,\mathbb{Q}}] \quad (\text{C.1})$$

Let us consider the \mathbb{Q} -Brownian motion W given by:

$$dW_t = \frac{dz_t^{X,\mathbb{Q}} - \rho_{FX} dz_t^{F,\mathbb{Q}}}{\sqrt{1 - \rho_{FX}^2}}$$

Levy's theorem shows that it is independent of $z^{F,\mathbb{Q}}$. We can rewrite (C.1) in terms of $z^{F,\mathbb{Q}}$ and W , for $t = T$:

$$e^{-rT}(X_T - K)^+ = c(0, F_0, X_0) + \int_0^T e^{-rt} [\hat{n}_t dF_t + c_X \sigma_X X_t dW_t]$$

This is to be compared to the following equality, which follows from the dynamics of V (see (8)):

$$e^{-rT}V_T = V_0 + \int_0^T e^{-rt} \hat{n}_t dF_t$$

The result follows, since as shown by Girsanov's theorem, W is also a Brownian motion under \mathbb{P} .

Appendix D: Expression for the Log-Likelihood

The state vector for the model of section 1 is $\mathbf{Z}_t = \begin{pmatrix} \ln F_t & \ln X_t \end{pmatrix}'$ and its dynamics under \mathbb{P} is given by:

$$d\mathbf{Z}_t = (\mathbf{B}^{\mathbb{P}} + \mathbf{A}_t^{\mathbb{P}} \mathbf{Z}_t) dt + \boldsymbol{\sigma}' d\mathbf{z}_t^{\mathbb{P}}$$

where:

$$\mathbf{B}^{\mathbb{P}} = \begin{pmatrix} \mu_S - \frac{\sigma_S^2}{2} \\ \mu_X - \frac{\sigma_X^2}{2} \end{pmatrix}, \quad \mathbf{A}_t^{\mathbb{P}} = \begin{pmatrix} -\frac{a}{T_0-t} & \frac{a}{T_0-t} \\ 0 & 0 \end{pmatrix}$$

and $\boldsymbol{\sigma}$ is a matrix such that:

$$\boldsymbol{\sigma}' \boldsymbol{\sigma} = \begin{pmatrix} \sigma_F^2 & \rho_{FX} \sigma_F \sigma_X \\ \rho_{FX} \sigma_F \sigma_X & \sigma_X^2 \end{pmatrix}$$

Integrating the dynamics of Z from t to $t+h$, we obtain a VAR representation for the model:

$$\mathbf{Z}_{t+h} = \boldsymbol{\Phi}_{0,t,h} + \boldsymbol{\Phi}_{1,t,h} \mathbf{Z}_t + \mathbf{U}_t^{\mathbb{P}}$$

where $\mathbf{U}_t^{\mathbb{P}}$ is normally distributed with mean vector $(0, 0)'$ and covariance matrix $\boldsymbol{\Sigma}_{t,h}^{\mathbb{P}}$. In details we have:

$$\boldsymbol{\Phi}_{0,t,h} = \int_0^h e^{\int_{t+h-u}^{t+h} \mathbf{A}_s^{\mathbb{P}} ds} \mathbf{B}^{\mathbb{P}} du, \quad \boldsymbol{\Phi}_{1,t,h} = e^{\int_t^{t+h} \mathbf{A}_s^{\mathbb{P}} ds}$$

and:

$$\boldsymbol{\Sigma}_{t,h}^{\mathbb{P}} = \int_0^h e^{\int_{t+h-u}^{t+h} \mathbf{A}_s^{\mathbb{P}} ds} \boldsymbol{\sigma}' \boldsymbol{\sigma} e^{\int_{t+h-u}^{t+h} (\mathbf{A}_s^{\mathbb{P}})' ds} du$$

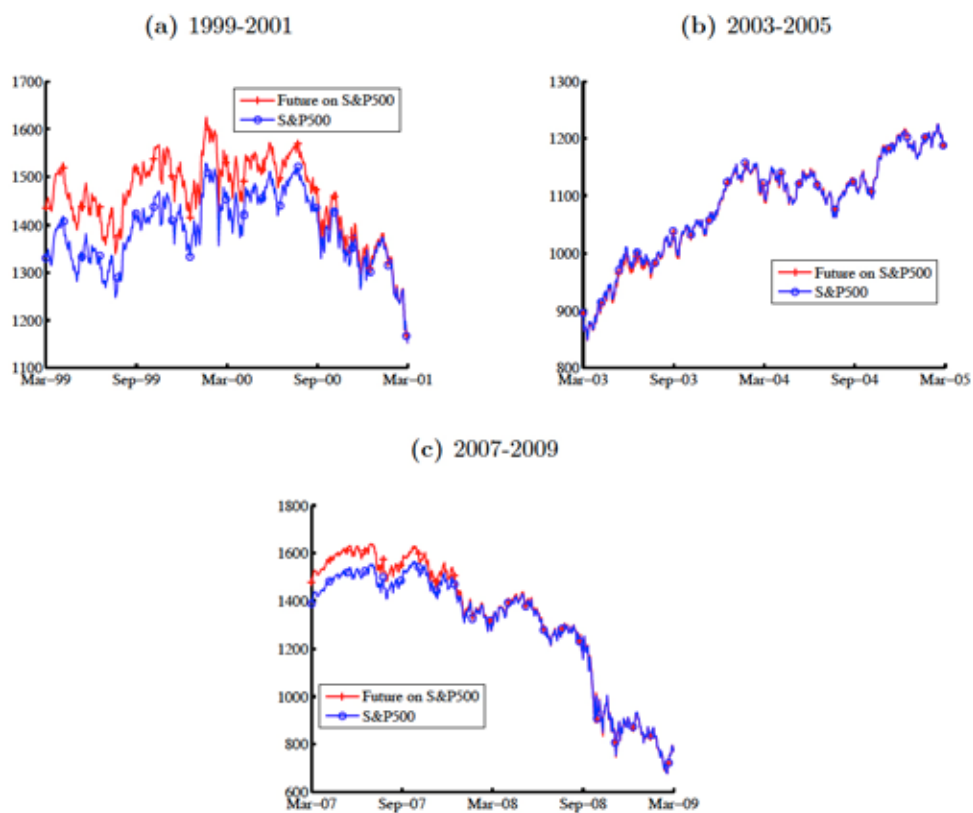
In particular, conditional on \mathbf{Z}_t , \mathbf{Z}_{t+h} follows a two-dimensional normal distribution with mean $\boldsymbol{\Phi}_{0,t,h} + \boldsymbol{\Phi}_{1,t,h} \mathbf{Z}_t$ and variance $\boldsymbol{\Sigma}_{t,h}^{\mathbb{P}}$. t_0, \dots, t_n are the dates where observations of Z are available and $\Delta t = t_{i+1} - t_i$ the (assumed constant) step between those dates. From Bayes' formula, the log-likelihood of these observations is:

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^n \mathcal{L}(\mathbf{Z}_{t_i} | \mathbf{Z}_{t_{i-1}}) \\ &= -\frac{1}{2} \sum_{i=1}^n [\mathbf{Z}_{t_i} - \boldsymbol{\Phi}_{0,t_{i-1},\Delta t} - \boldsymbol{\Phi}_{1,t_{i-1},\Delta t} \mathbf{Z}_{t_{i-1}}]' \left(\boldsymbol{\Sigma}_{t_{i-1},\Delta t}^{\mathbb{P}} \right)^{-1} [\mathbf{Z}_{t_i} - \boldsymbol{\Phi}_{0,t_{i-1},\Delta t} - \boldsymbol{\Phi}_{1,t_{i-1},\Delta t} \mathbf{Z}_{t_{i-1}}] \\ &\quad - \frac{1}{2} \sum_{i=1}^n \ln \det \boldsymbol{\Sigma}_{t_{i-1},\Delta t}^{\mathbb{P}} - \frac{n}{2} \ln 2\pi \end{aligned}$$

One can then maximize this log-likelihood function with respect to the parameters of the continuous-time model.

Figure and Tables

Figure 1: Time series for spot and futures prices.



These figures plot joint time series for the S&P 500 index and futures contracts written on the S&P 500 over three different two-year periods. All futures contracts have an initial maturity equal to two years.

Table 1: Calibrated parameter values.

(a) Drift restriction $\mu_X = \mu_S = 10\%$.

| Period | 98-00 | 99-01 | 00-02 | 01-03 | 02-04 | 03-05 | 04-06 | 05-07 | 06-08 | 07-09 |
|---|-------|--------|--------|-------|--------|--------|--------|--------|--------|--------|
| Panel A1: Constrained Values | | | | | | | | | | |
| μ_X | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 |
| Panel B1: Maximum Likelihood Estimates | | | | | | | | | | |
| a | 2.573 | 2.272 | 3.463 | 8.144 | 12.391 | 37.235 | 2.828 | 1.330 | 1.830 | 6.917 |
| σ_X | 0.201 | 0.195 | 0.210 | 0.239 | 0.216 | 0.126 | 0.105 | 0.104 | 0.152 | 0.337 |
| σ_F | 0.216 | 0.198 | 0.212 | 0.243 | 0.218 | 0.125 | 0.104 | 0.103 | 0.150 | 0.341 |
| ρ_{FX} | 0.973 | 0.965 | 0.967 | 0.983 | 0.984 | 0.975 | 0.973 | 0.979 | 0.982 | 0.983 |
| Panel C1: Closed-form Expressions | | | | | | | | | | |
| σ_D | 0.051 | 0.052 | 0.055 | 0.045 | 0.038 | 0.028 | 0.024 | 0.021 | 0.029 | 0.064 |
| ρ_{DX} | 0.166 | -0.082 | -0.086 | 0.014 | -0.051 | -0.148 | -0.144 | -0.135 | -0.193 | -0.028 |

(b) Drift restriction $\mu_X = \mu_S = r = 3\%$.

| Period | 98-00 | 99-01 | 00-02 | 01-03 | 02-04 | 03-05 | 04-06 | 05-07 | 06-08 | 07-09 |
|---|-------|--------|--------|-------|--------|--------|--------|--------|--------|--------|
| Panel A2: Constrained Values | | | | | | | | | | |
| μ_X | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 | 0.030 |
| Panel B2: Maximum Likelihood Estimates | | | | | | | | | | |
| a | 2.634 | 2.242 | 3.435 | 8.150 | 12.388 | 37.250 | 2.719 | 1.272 | 1.761 | 6.911 |
| σ_X | 0.201 | 0.195 | 0.210 | 0.238 | 0.216 | 0.126 | 0.105 | 0.104 | 0.152 | 0.337 |
| σ_F | 0.216 | 0.198 | 0.212 | 0.243 | 0.218 | 0.125 | 0.104 | 0.103 | 0.150 | 0.341 |
| ρ_{FX} | 0.973 | 0.965 | 0.967 | 0.983 | 0.984 | 0.975 | 0.973 | 0.979 | 0.982 | 0.982 |
| Panel C2: Closed-form Expressions | | | | | | | | | | |
| σ_D | 0.051 | 0.052 | 0.055 | 0.045 | 0.038 | 0.028 | 0.024 | 0.021 | 0.029 | 0.064 |
| ρ_{DX} | 0.168 | -0.080 | -0.084 | 0.016 | -0.051 | -0.147 | -0.140 | -0.133 | -0.191 | -0.028 |

These tables display parameter values calibrated to market data, over ten two-year rolling windows that cover the period 1998-2009. μ , σ and ρ respectively denote drift, volatility and correlation parameters. The indices X , F and D respectively refer to the underlying of the option, the futures price and the basis process defined as $D = \ln(F/X)$. a is the rate of convergence to zero in the dynamics of the basis process, and r is the risk-free rate. Parameter values in panels A1 and A2 have been exogenously imposed; those in panels B1 and B2 have been obtained through likelihood maximization and those in panels C1 and C2 have been obtained through equations (4) and (5). In table (a), both drifts have been set to 10%, and in table (b), they have been set to 3%.

Table 2: Base-case parameter values with basis risk only.

| | |
|--|----------|
| Underlying of the option | |
| Initial value (X_0) | 1 |
| Volatility (σ_X) | 0.1983 |
| Drift (μ_X) | 0.10 |
| Basis process | |
| Initial value (D_0) | 0.0125 |
| Rate of convergence to zero (a) | 3.1454 |
| Volatility (σ_D) | 0.0417 |
| Maturity of the futures contract (T_0) | 2 years |
| Correlations | |
| Underlying of the option with basis process (ρ) | -0.0839 |
| Option characteristics | |
| Risk-free rate (r) | 0.03 |
| Strike price (K) | 1 |
| Maturity (T) | 3 months |

This table displays the base-case parameters. The dynamics of the state variables X and D are given in equations (1) and (2).

Table 3: Absolute replication error of indifference hedging strategy (in %).

| $T_0 - T = 1 \text{ month}$ | | | | | | | | |
|------------------------------|------|------|------|------|------|------|------|------|
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.025 | 0.04 | 0.02 | 0.02 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| 0.05 | 0.14 | 0.09 | 0.06 | 0.04 | 0.03 | 0.03 | 0.02 | 0.02 |
| 0.075 | 0.32 | 0.20 | 0.13 | 0.10 | 0.07 | 0.06 | 0.05 | 0.05 |
| 0.1 | 0.56 | 0.35 | 0.24 | 0.17 | 0.14 | 0.11 | 0.09 | 0.08 |
| $T_0 - T = 3 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.025 | 0.05 | 0.04 | 0.03 | 0.03 | 0.02 | 0.02 | 0.02 | 0.01 |
| 0.05 | 0.20 | 0.16 | 0.12 | 0.10 | 0.08 | 0.07 | 0.06 | 0.05 |
| 0.075 | 0.45 | 0.35 | 0.28 | 0.22 | 0.19 | 0.16 | 0.14 | 0.12 |
| 0.1 | 0.76 | 0.60 | 0.48 | 0.40 | 0.33 | 0.28 | 0.25 | 0.22 |
| $T_0 - T = 6 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.025 | 0.06 | 0.06 | 0.05 | 0.05 | 0.04 | 0.04 | 0.04 | 0.03 |
| 0.05 | 0.25 | 0.22 | 0.20 | 0.18 | 0.17 | 0.15 | 0.14 | 0.13 |
| 0.075 | 0.53 | 0.48 | 0.44 | 0.40 | 0.36 | 0.33 | 0.31 | 0.28 |
| 0.1 | 0.89 | 0.81 | 0.75 | 0.68 | 0.63 | 0.58 | 0.53 | 0.49 |

This table displays the square root of quadratic replication error, $\sqrt{R_0}$, for the indifference hedging strategy. σ_D is the volatility of the log difference between the future price F and the spot price X ; a is the speed of mean reversion of the basis process towards zero; T_0 and T are respective maturities of the futures contract and the option. Other parameters are fixed at their base-case values (see table 2).

Table 4: Relative replication error of indifference hedging strategy (in % of the reservation price).

| $T_0 - T = 1 \text{ month}$ | | | | | | | | |
|------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.025 | 14.61 | 11.02 | 8.99 | 7.77 | 6.97 | 6.43 | 6.02 | 5.72 |
| 0.05 | 28.64 | 21.46 | 17.26 | 14.66 | 12.94 | 11.73 | 10.84 | 10.14 |
| 0.075 | 42.30 | 31.93 | 25.66 | 21.69 | 19.04 | 17.16 | 15.76 | 14.68 |
| 0.1 | 55.12 | 42.15 | 33.96 | 28.64 | 25.03 | 22.46 | 20.55 | 19.07 |
| $T_0 - T = 3 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.025 | 17.46 | 14.74 | 12.76 | 11.29 | 10.19 | 9.34 | 8.67 | 8.14 |
| 0.05 | 34.20 | 28.91 | 25.00 | 22.05 | 19.80 | 18.05 | 16.66 | 15.54 |
| 0.075 | 50.04 | 42.67 | 37.07 | 32.78 | 29.44 | 26.81 | 24.71 | 23.01 |
| 0.1 | 64.36 | 55.54 | 48.61 | 43.16 | 38.85 | 35.40 | 32.60 | 30.32 |
| $T_0 - T = 6 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.025 | 19.22 | 17.48 | 16.02 | 14.77 | 13.70 | 12.78 | 11.98 | 11.29 |
| 0.05 | 37.56 | 34.27 | 31.45 | 29.04 | 26.95 | 25.14 | 23.56 | 22.18 |
| 0.075 | 54.61 | 50.13 | 46.24 | 42.86 | 39.91 | 37.32 | 35.04 | 33.04 |
| 0.1 | 69.65 | 64.44 | 59.84 | 55.77 | 52.16 | 48.96 | 46.11 | 43.57 |

This table displays the square root of quadratic replication error, $\sqrt{R_0}$, for the indifference hedging strategy, in % of the reservation price. D is the volatility of the log difference between the future price F and the spot price X ; a is the speed of mean reversion of the basis process towards zero; T_0 and T are respective maturities of the futures contract and the option. Other parameters are fixed at their base-case values (see table 2).

Table 5: Indifference hedging strategy vs. Black strategy – Percentage of increase in replication error.

| $T_0 - T = 1 \text{ month}$ | | | | | | | | |
|------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 75.87 | 39.06 | 21.74 | 13.89 | 10.11 | 8.02 | 6.67 | 5.69 |
| 0.05 | 47.60 | 27.68 | 17.45 | 12.16 | 9.22 | 7.42 | 6.19 | 5.30 |
| 0.075 | 42.90 | 23.91 | 14.20 | 9.23 | 6.56 | 5.00 | 4.00 | 3.30 |
| 0.1 | 42.81 | 21.60 | 10.94 | 5.80 | 3.29 | 2.00 | 1.27 | 0.82 |
| $T_0 - T = 3 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 183.96 | 122.90 | 82.17 | 55.54 | 38.49 | 27.75 | 21.04 | 16.81 |
| 0.05 | 85.04 | 60.39 | 43.80 | 32.66 | 25.18 | 20.15 | 16.72 | 14.33 |
| 0.075 | 70.18 | 50.26 | 36.78 | 27.64 | 21.42 | 17.14 | 14.14 | 12.00 |
| 0.1 | 69.62 | 49.08 | 35.26 | 25.85 | 19.38 | 14.90 | 11.81 | 9.63 |
| $T_0 - T = 6 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 814.57 | 669.36 | 550.33 | 452.35 | 371.51 | 304.76 | 249.70 | 204.36 |
| 0.05 | 260.96 | 215.95 | 179.25 | 149.16 | 124.42 | 104.05 | 87.27 | 73.47 |
| 0.075 | 165.81 | 137.77 | 114.97 | 96.37 | 81.13 | 68.60 | 58.30 | 49.81 |
| 0.1 | 141.11 | 117.35 | 98.06 | 82.34 | 69.47 | 58.91 | 50.21 | 43.03 |

This table shows the percentage of increase in the square root of quadratic replication error, $\sqrt{R_0}$, when replicating the option on X by using the Black formula. σ_D is the volatility of the log difference between the future price F and the spot price X ; a is the speed of mean reversion of the basis process towards zero; T_0 and T are respective maturities of the futures contract and the option. Other parameters are fixed at their base-case values (see table 2).

Table 6: Impact of an estimation error in the drift of the underlying asset of the option – Percentage of increase in replication error.

| $T_0 - T = 1 \text{ month}$ | | | | | | | | |
|------------------------------|------|------|------|------|------|------|------|------|
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 0.05 | 0.05 | 0.05 | 0.04 | 0.03 | 0.03 | 0.02 | 0.02 |
| 0.05 | 0.27 | 0.29 | 0.28 | 0.24 | 0.20 | 0.17 | 0.14 | 0.11 |
| 0.075 | 0.57 | 0.64 | 0.63 | 0.56 | 0.47 | 0.40 | 0.33 | 0.28 |
| 0.1 | 0.91 | 1.05 | 1.05 | 0.96 | 0.84 | 0.72 | 0.61 | 0.52 |
| $T_0 - T = 3 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.04 |
| 0.05 | 0.24 | 0.27 | 0.29 | 0.30 | 0.30 | 0.29 | 0.27 | 0.26 |
| 0.075 | 0.50 | 0.58 | 0.63 | 0.65 | 0.66 | 0.64 | 0.62 | 0.58 |
| 0.1 | 0.79 | 0.92 | 1.01 | 1.06 | 1.07 | 1.07 | 1.04 | 0.99 |
| $T_0 - T = 6 \text{ months}$ | | | | | | | | |
| $\sigma_D \backslash a$ | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| 0 | – | – | – | – | – | – | – | – |
| 0.025 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 0.05 | 0.21 | 0.23 | 0.25 | 0.26 | 0.28 | 0.29 | 0.29 | 0.30 |
| 0.075 | 0.45 | 0.49 | 0.53 | 0.56 | 0.59 | 0.61 | 0.63 | 0.65 |
| 0.1 | 0.70 | 0.77 | 0.83 | 0.88 | 0.93 | 0.98 | 1.01 | 1.04 |

This table shows the percentage of increase in the square root of quadratic replication error, $\sqrt{R_0}$, when the indifference hedging strategy of proposition 2 is implemented under the assumption that the drift X equals the risk-free rate. D is the volatility of the log difference between the future price F and the spot price X ; a is the speed of mean reversion of the basis process towards zero; T_0 and T are respective maturities of the futures contract and the option. Other parameters are fixed at their base-case values (see table 2).

References

- Bernardo, A. and O. Ledoit (2000). Gain, loss, and asset pricing. *Journal of Political Economy* 108 (1), 144-172.
- Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics* 3 (1-2), 167-179.
- Cochrane, J. and J. Saa-Requejo (2000). Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy* 108 (1), 79-119.
- Collin-Dufresne, P. and J. Hugonnier (2007). Pricing and hedging in the presence of extraneous risks. *Stochastic Processes and their Applications* 117 (6), 742-765.
- Davis, M. (2006). Optimal hedging with basis risk. *From Stochastic Calculus to Mathematical Finance*, 169-187.
- Detemple, J. and S. Sundaresan (1999). Nontraded asset valuation with portfolio constraints: A binomial approach. *Review of Financial Studies* 12 (4), 835-872.
- Duffie, D. (2001). *Dynamic Asset Pricing Theory* (Third ed.). Princeton University Press, NJ.
- Duffie, D., W. Fleming, H. Soner, and T. Zariphopoulou (1997). Hedging in incomplete markets with HARA utility. *Journal of Economic Dynamics and Control* 21 (4-5), 753-782.
- Duffie, D. and H. Richardson (1991). Mean-variance hedging in continuous time. *The Annals of Applied Probability* 1 (1), 1-15.
- Dybvig, P. (1992). Hedging non-traded wealth: When is there separation of hedging and investment? *Options: Recent Advances in Theory and Practice* 2, 13-24.
- Föllmer, H. and M. Schweizer (1990). Hedging of contingent claims under incomplete information. pp. 389-414.
- He, H. and N. D. Pearson (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. *Journal of Economic Theory* 54 (2), 259-304.
- Henderson, V. (2002). Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance* 12 (4), 351-373.
- Hodges, S. and A. Neuberger (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets* 8 (2), 222-239.
- Karatzas, I. and S. Shreve (1991). *Brownian motion and stochastic calculus*. Springer.
- Lioui, A. and P. Poncet (2000). The minimum variance hedge ratio under stochastic interest rates. *Management Science* 46 (5), 658-668.
- Liu, J. and F. Longsta (2004). Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities. *Review of Financial Studies* 17 (3), 611-641.
- Merton, R. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics* 8 (4), 323-361.
- Monoyios, M. (2004). Performance of utility based strategies for hedging basis risk. *Quantitative Finance* 4(3), 245-255.
- Musiela, M. and T. Zariphopoulou (2004). An example of indifference prices under exponential preferences. *Finance and Stochastics* 8 (2), 229-239.
- Rouge, R. and N. El Karoui (2000). Pricing via utility maximization and entropy. *Mathematical Finance* 10 (2), 259-276.

- Schweizer, M. (1991). Option hedging for semimartingales. *Stochastic Processes and their Applications* 37 (2), 339-363.
- Schweizer, M. (2001). A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, M. Musiela (Eds), *Option pricing, interest rates and risk management*, pp. 538-574.
- Soner, H., S. Shreve, and J. Cvitanic (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability* 5 (2), 327-355.
- Tehranchi, M. (2004). Explicit solutions of some utility maximization problems in incomplete markets. *Stochastic Processes and Their Applications* 114 (1), 109-125.
- Zariphopoulou, T. (2001). A solution approach to valuation with unhedgeable risks. *Finance and Stochastics* 5 (1), 61-82.