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Portfolio Choice for Oil-Based Sovereign Wealth Funds

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Abstract

Given recent interest in the activities of sovereign wealth funds (SWFs), this paper will review the financial economics of portfolio choice for oil-based investors. We view the optimal asset allocation problem of a sovereign wealth fund as the decision-making problem of an investor with non-tradable endowed wealth (oil reserves). Optimal portfolios combine speculative demand (optimal growth) as well as hedging demand (hedging resource fluctuation risk) and the level of risk taking should depend both on the fraction of financial wealth to resource wealth and on the oil shock hedging properties of the investments. As a novelty in the theoretical literature we introduce background risk for an SWF in the form of oil reserve uncertainty. SWFs with great uncertainty about the size of their reserves should invest less aggressively and vice-versa. We also identify the optimal speed of the extraction policy (oil-to-equity transformation) as a driving force for portfolio adjustments across time and present a dynamic programming approach to approximate portfolio adjustments.

1. Introduction

For the purpose of this paper we define sovereign wealth funds (SWFs) as sovereign investment vehicles (returns enter the government's fiscal budget) with high foreign asset exposure, non-standard liabilities and a long (intergenerational) time horizon.[1] In this paper we focus on SWFs sourced by oil revenues as the currently most important (largest) fraction of this class of new investors (table 1).

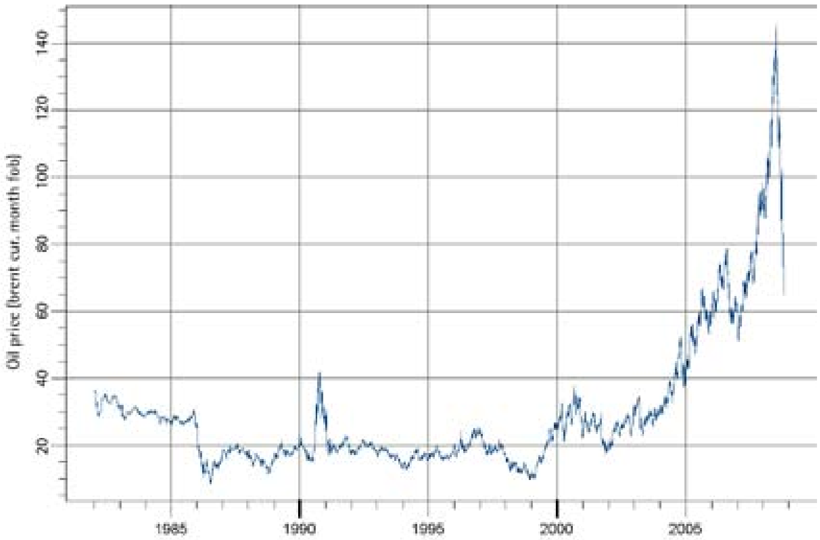
Table 1. The ten biggest SWFs: size and source of funding. All numbers USD billion and based on public sources or our own estimates as of the end of 2007.

Sovereign	Assets	Inception	Source	Weight
UAE	880	1976	Oil	29.33%
Norway	390	1996	Oil	13.00%
Singapore	350	1981	Misc	11.67%
Saudi Arabia	290	1981	Oil	9.67%
Kuwait	245	1953	Oil	8.17%
China	200	2007	Misc	6.67%
Lybia	55	1974	Oil	1.83%
Qatar	49	N/A	Oil	1.63%
Algeria	44	2000	Oil	1.47%
USA (Alaska)	39	1976	Oil	1.30%

Eight of the ten biggest SWFs are sourced from oil revenues. Given an estimated market size of about USD3 trillion at the beginning of 2008, the three biggest oil revenue funds account for 52% of total SWF assets. Given the mediocre long-term performance of spot oil (underground wealth), SWFs were created to effect an oil-to-equity transformation to participate in global growth. The speed of this transformation will depend on the optimal path of extraction, which depends on the impact of increased supply on oil prices, extraction costs (technology) and oil price expectations. Given an estimated USD 40 trillion value of underground oil compared to USD50 trillion in global equities, SWFs will have a major impact on global equity markets. It will also lead to a shift from traditional reserve currencies (US dollars, Japanese yen) to the currencies of emerging markets, where much of the global growth is to be expected.

For many oil-exporting countries, crude oil or gas reserves are the single most important national asset. Any change in the value of reserves directly and materially affects these countries' wealth, and thus the wellbeing of their citizens. Figure 1 serves as an illustration. Oil prices fluctuate wildly, and these fluctuations can destabilize the economy via volatile real exchange rates.

Figure 1. Daily oil price movements from January 1982 to September 2008. The underlying total wealth position of an oil-rich country can vary dramatically over time and needs management to smooth intergenerational consumption patterns.



1 - A long time horizon does not imply low risk aversion. This is one of the most common fallacies made in asset management and usually rests with the focus on quantile-based risk management.

Having recognized this, a number of oil-exporting countries have been depositing oil revenues in funds dedicated to future expenditure. Devising optimal investment policies for such oil revenue funds is the aim of this paper. We analyze optimal allocations among standard partitions of the investment universe, taking into account that aggregate wealth consists of financial assets and oil reserves.

An example of an oil revenue fund is Norway's State Petroleum Fund. The policy goals of the fund, as stated in the Norske Finansdepartementet's (Norwegian Ministry of Finance) Summary,[2] is "first, [to] act as a buffer to smooth short term variations in the oil revenues [in the Fiscal Budget, ... and second to] serve as a tool for coping with the financial challenges connected to an ageing population and the eventual decline in oil revenues, by transferring wealth to future generations." The second objective is "invest[ing] the capital in such a way that the fund's international purchasing power is maximized, taking into account an acceptable level of risk." This suggests that the benchmark of the fund is future consumption in the form of imports. The same reason also motivates the inclusion of equity, which is expected to enhance the performance of the fund. Concerning the definition of risk, it appears that the Finansdepartementet is mostly concerned with changes in market value of the fund. We were not able to infer the Finansdepartementet's views on achieving the first objective, smoothing oil revenues in the short term. We believe that both objectives, smoothing revenues and maximizing long-term welfare, suggest the more extensive definition of risk we propose in this paper.[3]

More generally, our paper is an example of how risk stemming from non-financial assets can be hedged, at least partially, through financial assets. In other words: we talk about asset allocation with non-tradable wealth. The key is exploiting the correlation between financial and non-financial assets to reduce the overall risk of the portfolio, compared to an allocation that considers only the correlation structure of the financial assets. Although the general idea is straightforward, empirical or practical implementations are rare. An exception is asset/liability management, in which interest rate exposure on one side of the balance sheet is offset by interest rate exposure on the other side. This paper applies a similar idea to a more general problem.

We will focus on portfolio investments. This is a narrower brief than what sovereign investors can do. Rather than investing in securities (mostly USD-dominated) abroad, sovereign investors can also use their oil revenues to build exposure to future growth industries and develop the necessary infrastructure to make their country an attractive place to attract top human talent. Dubai and Qatar are prime examples.

During the following exposition we will rely on the assumption of normality in return distributions to allow us to come up with closed-form solutions that provide conceptual insight into the structure of the underlying problem. While we are aware that returns on capital markets and certainly on commodities like oil are in the short run far from normal, we also believe an SWF belongs to the group of long-term investors such that the central limit theorem will come to our help to somewhat mitigate the non-normality issue.

The outline is as follows. Section 2 describes the general portfolio choice problem for an oil (commodity) based SWF together with some empirical evidence on the oil shock hedging properties for different investments. Section 3 extends section 2 by differentiating between hedging and growth assets. In section 4 we introduce background risk into the SWF asset allocation problem, by making the level of resource wealth a random variable. Section 5 solves the inter-temporal asset allocation problem and section 6 concludes.

2 - Http://www.odin.dep.no/fin/engelsk/p10001617/p10002780/indexbna.html. Further information regarding the aims and policies of the fund is in the Annual Reports, Kjaer (2001), and Norges Bank (2002).

3 - Other examples of portfolios funded by revenues from natural resources include the Alaska Permanent Reserve Fund (funds of USD 23 billion), the State Oil Fund of Azerbaijan (USD 0.5 billion), Chad's Revenue Management Fund, the National Fund of Kazakhstan (USD 1.2 billion), Venezuela's Investment Fund for Macroeconomic Stabilization (USD 3.7 billion), the Alberta Heritage Savings Trust Fund (CAD 3.7 billion), and the Nunavut Trust (CAD 0.5 billion). Furthermore, certain central bank funds of oil exporting countries, such as Iran, Kuwait, Oman, and Saudi Arabia, are de-facto oil revenue funds. In general, stated investment objectives are similar to those of the Norwegian fund; *i.e.*, a favorable long-term tradeoff of return and risk of the financial portfolio. The risk in aggregate wealth stemming from price changes in natural reserves is typically ignored.

2. Incorporating the Sovereign Wealth Fund into Government Budgets

We view the optimal asset allocation problem of an SWF as the decision-making problem of an investor with non-tradable endowed wealth (oil reserves). In order to get insight into the portfolio choice problem for an SWF we assume the following analytical setup.

The SWF can invest its financial wealth in a single asset or cash. We can think of this as the choice between the global market portfolio and cash. This is certainly restrictive, but will allow us to develop our framework without the need for very complex calculations and we will relax this assumption in the following section. Returns for this performance asset are normally distributed and given by

(1) $$\tilde{r}_a \sim N\left(\mu_a, \sigma_a^2\right)$$

where μ_a represents the expected risk premium (over local cash returns) of our performance asset and σ_a its volatility. At the same time, the government budget moves with changes on its claim on economic net wealth. For a commodity- (oil-) based SWFs changes in commodity (oil) prices will have the greatest influence by far on the government budget measured in economic (not accounting) terms. We assume that oil price changes are also normally distributed

(2) $$\tilde{r}_o \sim N\left(\mu_o, \sigma_o^2\right)$$

and correlate positively with asset returns, *i.e.*, $Cov(\tilde{r}_a, \tilde{r}_o) = \rho_{a,o} > 0$. As μ_o is empirically extremely noisy to estimate we look for an economic prior. Given that under perfectly integrated capital markets the Hotelling-Solow rule states that natural resource prices should grow at the world interest rate such that countries are indifferent between depletion (earning the interest rate) and keeping oil underground (earning price changes). We hence assume a risk premium on oil of zero, *i.e.*, $\mu_o = 0$. Brent prices on 4 January 1982 were 35.9 and rose to 66.6 on 21 October 2008. This amounts to a meager 2.3% return per annum over the last 26.8 years, which makes our assumption of a zero risk premium on underground oil suddenly look much more realistic. Even if we used the maximum oil price of 145.61 this would amount to a mere 5.3% return, which is even more in line with average money market returns.

How do we integrate oil wealth into a country's budget surplus (deficit)? Let θ denote the fraction of importance the SWF plays in the government budget. A simple way to gauge this is the following consideration. If the SWF has a size of 1 monetary unit, while the market value of oil reserves amounts to 5 monetary units, this translates into a $\theta = \frac{1}{1+5} = \frac{1}{6}$ weight for the SWF asset and $1-\theta = 1-\frac{1}{6} = \frac{5}{6}$ weight for oil revenues.[4]

In other words

(3) $$\tilde{r} = \theta w \tilde{r}_a + (1-\theta)\tilde{r}_o$$

Note that $1-\omega$ represents the implied cash holding that carries a zero risk premium and no risk in a one-period consideration. Expressing returns as risk premium has the advantage that we do not need to model cash holdings. These simply become the residual asset that ensures portfolio weights add up to one without changing risk or (excess) return.

Suppose now the SWF manager is charged with maximizing the utility of total government wealth rather than narrowly maximizing the utility for direct assets under management. The optimal solution for this problem can be found from

(4) $$\max_w \left(\theta w \mu_a - \frac{\lambda}{2}\left[\theta^2 w^2 \sigma_a^2 + (1-\theta)^2 \sigma_o^2 + 2w\theta(1-\theta)\rho\sigma_a\sigma_o\right]\right)$$

4 - Alternatively, one could incorporate all other items (tax revenues, government spending, pension liabilities, etc.) into the government budget surplus/deficit calculation as in Doskeland (2007). However, given the relatively low volatility of these positions, we will ignore this problem. It will not change the nature of our findings.

Taking first-order conditions and solving for ω we arrive at the optimal asset allocation for a resource-based SWF.

(5)
$$w^* = w_s^* + w_h^* = \frac{1}{\theta}\frac{\mu_a}{\lambda\sigma_a^2} - \frac{1-\theta}{\theta}\frac{\rho\sigma_o}{\sigma_a}$$

Total demand for risky assets can be decomposed into speculative demand w_s^* and hedging demand, w_h^*. In the case of uncorrelated assets and oil resources the optimal solution is equivalent to a leveraged (with factor $\frac{1}{\theta}$) position in the asset only maximum Sharpe-ratio portfolio or in other words w_s^*. What is the economic intuition for this leverage? For investors with constant relative risk aversion the optimal weight of risky assets will be independent of wealth, which nowhere enters (4). While a given country might have little in financial wealth in the form of SWF financial assets it might be rich in natural resources and as such it requires a large multiplier. For $\theta = \frac{1}{6}$ we would require the SWF to leverage substantially (six times). Assuming $\mu_a = 5, \sigma_a = 20, \lambda = 0.03$, we get $\frac{1}{\theta}\frac{6}{0.03 \cdot 20^2} = 250\%$. The second component in (5) represents hedging demand. In other words: the desirability of the risky asset does depends not only on the Sharpe ratio but also on its ability to hedge out unanticipated shocks to oil wealth. Hedging demand is given as the product of leverage and oil asset beta, $\beta_{o,a} = \frac{\rho\sigma_o}{\sigma_a}$. The latter is equivalent to the slope coefficient of a regression of (demeaned) asset returns against (demeaned) oil returns, *i.e.*, of the form

(6)
$$(r_o - \bar{r}_o) = \beta_{o,a}(r_a - \bar{r}_a) + \varepsilon$$

Hedge demand is zero oil if oil price risk is purely idiosyncratic. For $\sigma_o = 40, \rho = 0.1$, we would reduce the allocation in the risky asset according to $-\frac{1-\frac{1}{6}}{\frac{1}{6}}\frac{0.1 \cdot 40}{20} = -100\%$ Positive correlation of asset and oil price risk increases the volatility of total wealth. A 100% short position in the risky asset helps to mange total risk. However, if the correlation was negative we would even further increase the allocation to the risky asset. The optimal position of the SWF would be 1.5 times leverage in the global market portfolio.

While the focus of this paper is not on empirical work, we should provide some indication on the oil shock hedging properties of traditional asset classes. Without the existence of these assets that could potentially help to reduce total wealth volatility for oil-rich investors, equation (5) would be of little practical use. Let us look at global equities and (MSCI World in USD) and US government bonds (Lehman US Treasury total return index for varying maturities) and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1997 to September 2008. The selection of the above-mentioned assets is motivated by some basic economic considerations. Oil tends to do well either in a political crisis (in which equities do not do well) or in anticipation of global growth (in which equities also do well). At the same time, government bonds (particular at the long end) are a natural recession hedge and will also do well if oil prices fall. There are obviously notable exceptions. Oil and bonds will move together if an oil price increase is the cause of recession fears. In this scenario shorter bonds should provide better returns than long bonds due to rising inflation fears. All asset classes move together if monetary loosening creates a leverage-driven bubble that drives equity and bond markets, while bonds perform due to falling interest rates.

Table 2. Correlation of asset returns with percentage oil price changes. The table uses global equities (MSCI World in USD) and US government bonds (Lehman US Treasury total return index for varying maturities) and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1997 to September 2008. This translates into 142 monthly, 48 quarterly and 13 annual data points. For each data frequency, the first line shows the correlation coefficient, while the second line provides its *t*-value. We calculate *t*-values according to $t = \rho\sqrt{\frac{n-2}{1-\rho^2}}$,

where *n* represents the number of data points and ρ the estimated correlation coefficient. Critical values are given by the *t*-distribution with *n - 2* degrees of freedom. For example, the critical value for 13 annual data points at the 95% level is 2.2. All significant correlation coefficients are gray shaded.

	US Treasury Bond							Global
Frequency	1 year	1-3 year	3-5 year	5-7 year	7-10 year	10-20 year	20 plus year	Equities
Monthly	-8.28%	-2.38%	1.16%	2.26%	2.75%	1.18%	0.16%	9.19%
	-0.99	-0.43	0.21	0.41	0.49	0.21	0.03	1.66
Quarterly	-9.29%	-22.59%	-20.08%	-21.00%	-19.60%	-23.92%	-24.42%	-9.34%
	-0.65	-2.42	-2.14	-2.24	-2.09	-2.57	-2.63	-0.98
Annual	-38.81%	-40.70%	-50.94%	-56.52%	-59.48%	-55.04%	-51.29%	24.10%
	-1.52	-2.36	-3.13	-3.63	-3.92	-3.49	-3.16	1.31

Table 3. Correlation of US Industry returns with percentage oil price changes. The table uses Dow Jones sector returns and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1982 to September 2008. This translates into 323 monthly, 109 quarterly and 28 annual data points. For each data frequency, the first line shows the correlation coefficient, while the second line provides its *t*-value. We calculate *t*-values according to $t = \rho\sqrt{\frac{n-2}{1-\rho^2}}$, where *n* represents the number of data points and ρ the estimated correlation coefficient. Critical values are given by the *t*-distribution with *n - 2* degrees of freedom. For example, the critical value for 28 annual data points at the 95% level is 2.05. All significant correlation coefficients are gray shaded.

	Dow Jones Industries									
Frequency	Oil & Gas	Basic Mats	Industrials	Consumer Gds	Health Care	Consumer Svs	Telecom	Utilities	Financials	Technology
Monthly	45.68%	5.94%	-12.89%	-18.03%	-13.83%	-21.26%	-1.62%	-6.32%	-18.08%	-11.86%
	9.23	1.07	-2.34	-3.29	-2.51	-3.91	-0.29	-1.14	-3.30	-2.15
Quarterly	49.57%	8.11%	-6.47%	-4.17%	-28.57%	-22.69%	-4.55%	-6.20%	-24.99%	-25.03%
	5.96	0.85	-0.68	-0.44	-3.11	-2.43	-0.48	-0.65	-2.69	-2.70
Annual	51.03%	-20.03%	-11.99%	-65.36%	-37.26%	-51.40%	-44.18%	-15.84%	-13.52%	-3.47%
	3.14	-1.08	-0.64	-4.57	-2.12	-3.17	-2.61	-0.85	-0.72	-0.18

However, we can talk endlessly about our economic priors, so we should have a look at the data instead. The results of our correlation analysis are given in table 2. In the short term (monthly data) we do not find significant correlations between oil price change and the selected asset class return. However, reducing the data frequency, *i.e.*, increasing the period to calculate returns from, shows significantly negative correlations between oil price changes and fixed-income returns. In other words, we find that long-term correlations are drowned out by short-term noise.

Both the degree of (negative) correlation and its significance (even though we reduce the sample size) rise as we move from quarterly to annual. Global equities, however, provide no hedge against oil price changes. While they could still be used as a performance asset, they are of limited use as a hedge against oil price shocks.

Proponents of equity investments might suspect that we are underselling their case, as we have not been allowing for more granular equity exposures. We can perhaps identify sectors that respond differently to oil price shocks. A global equity portfolio is already a diversified portfolio that leaves no possibility to leverage these effects. The results for this can be found in table 3. Our results are encouraging. We find significant negative correlation for defensive consumer and health care sectors that tend to do well when the economy does badly. Results are stable and significant for different data frequencies. At the same time, the energy sector is positively related to oil and does not qualify for inclusion in SWF allocations as we would have conjectured before.

3. Optimal Allocation to Growth and Hedge Assets

The previous section has shown that traditional equities offer little protection from oil price risks. At the same time, fixed-income investments do, but they do not offer the same long-term returns and an SWF manager therefore needs to extend the investment universe to performance as well as hedge assets. We keep the setup from the previous section but extend the universe into two assets, where one asset is assumed a hedging asset (*i.e.*, it shows negative correlation), while the second asset provides growth orthogonal to oil wealth changes. We can summarize our setup with the following distribution

(7) $$\tilde{r}_g \sim N\left(\mu_g, \sigma_g^2\right), \quad \tilde{r}_h \sim N\left(\mu_h, \sigma_h^2\right)$$

where r_g and r_h stand for the return of growth and hedge assets with $\mu_g > \mu_h$. Our correlation assumptions are

(8) $$Cov\left(r_h, r_g\right) = \rho_{h,g}\sigma_h\sigma_g > 0, Cov\left(r_h, r_o\right) = \rho_{h,o}\sigma_h\sigma_o < 0, Cov\left(r_g, r_o\right) = 0$$

Our setup effectively assumes that an SWF can use leverage, but will only go long assets. The hedge asset is negatively correlated and as such will be held in non-negative demand.

This is necessary, as otherwise the assumption of non-tradable wealth would become meaningless. The government budget evolves according to

(9) $$\tilde{r} = \theta\left[w_g\tilde{r}_g + w_h\tilde{r}_h\right] + (1-\theta)\tilde{r}_o$$

where utility (*i.e.*, risk-adjusted performance) is given by

(10) $$u = E(\tilde{r}) - \tfrac{\lambda}{2}\left[E\left(\tilde{r}^2\right) - E(\tilde{r})^2\right]$$

We maximize (9) by setting the first-order conditions to zero and solving for w_g, w_h :

(11) $$w_g^* = \frac{\mu_g - \beta_{g,h}\cdot\mu_h}{\lambda\theta\left(1-\rho_{g,h}^2\right)\sigma_g^2} - \frac{(1-\theta)}{\theta}\frac{\beta_{g,h}\rho_{h,o}\sigma_o\sigma_h}{\left(1-\rho_{g,h}^2\right)}$$

(12) $$w_h^* = \frac{\mu_h - \beta_{h,g}\cdot\mu_g}{\lambda\theta\left(1-\rho_{g,h}^2\right)\sigma_h^2} - \frac{(1-\theta)}{\theta}\frac{\beta_{o,h}}{\left(1-\rho_{g,h}^2\right)}$$

where $\beta_{g,h} = \frac{\rho_{g,h}\sigma_g}{\sigma_h}, \beta_{o,h} = \frac{\rho_{h,o}\sigma_o}{\sigma_h}$. Demand for the growth asset can be split again into speculative demand and hedging demand. Speculative demand will depend on its "alpha", $\mu_g - \beta_{g,h}\cdot\mu_h$, versus the hedge asset, *i.e.*, "beta", $\beta_{g,h}$, adjusted excess return divided by the risk not explained by the hedge asset. Here, $\rho_{g,h}^2$ can be interpreted as the R^2 of a regression of hedge versus growth asset returns. Hedging demand in turn will depend on the implicit hedging through the correlation to the hedge asset. The usual risk aversion and oil wealth importance scaling applies. A similar picture is given for the hedge asset. However, the effect here is much more direct, such that the hedge demand will always be greater than for the growth asset (by definition). A clearer picture arises when we get rid of the indirect correlation by setting $\rho_{g,h} = 0$. In this case, (11) and (12) become

(13) $$w_g^* = \frac{\mu_g}{\lambda\theta\sigma_g^2}$$

(14) $$w_h^* = \frac{\mu_h}{\lambda\theta\sigma_h^2} - \rho_{h,o}\frac{\sigma_o}{\sigma_h}\frac{(1-\theta)}{\theta}$$

Now both allocations can be taken independently. The growth asset is entirely driven by its Sharpe ratio, while the hedge asset combines both speculative and hedge demand directly relating to (5). Finally, we will ask ourselves: how will the hedge demand (allocation of long-term USD-denominated fixed-income bonds) change as θ becomes larger, *i.e.*, as financial wealth becomes more and more dominant? In order to answer this question we calculate $\frac{dw_h^*}{d\theta}$ from (14).

Table 4: Correlation of hedge fund returns with percentage oil price changes. The table uses HFR index returns and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1997 to September 2008. This translates into 142 monthly, 48 quarterly and 13 annual data points. For each data frequency, the first line shows the correlation coefficient, while the second line provides its *t*-value. We calculate *t*-values according to $t = \rho\sqrt{\frac{n-2}{1-\rho^2}}$,
where n represents the number of data points and ρ the estimated correlation coefficient. Critical values are given by the *t*-distribution with $n - 2$ degrees of freedom. For example, the critical value for 13 annual data points at the 95% level is 2.2. All significant correlation coefficients are gray shaded.

	HFR Indices									
Frequency	Distressed Debt	Merger Arbitrage	Equity Market Neutral	Quantitative Directional	Short Bias	Event Driven	Global Macro	Relative Value	Fixed Income Arbitrage	Convertible Arbitrage
Monthly	10.50%	9.41%	8.56%	19.65%	-15.19%	13.28%	23.66%	13.02%	13.39%	7.61%
	1.26	1.13	1.02	2.39	-1.83	1.60	2.90	1.56	1.61	0.91
Quarterly	-7.64%	-19.83%	-15.28%	-12.01%	12.54%	-15.15%	-17.34%	-4.22%	14.12%	-10.86%
	-0.53	-1.40	-1.07	-0.84	0.88	-1.06	-1.22	-0.29	0.99	-0.76
Annual	20.38%	6.78%	-22.43%	44.78%	-33.86%	26.77%	41.76%	28.13%	30.09%	-0.88%
	0.75	0.24	-0.83	1.81	-1.30	1.00	1.66	1.06	1.14	-0.03

(15)
$$\frac{dw_h^*}{d\theta} = \frac{-\mu_h + \lambda\rho_{h,o}\sigma_h\sigma_o}{\lambda\sigma_h^2\theta^2} < 0$$

under the assumption that μ_h , $\mu_h > 0, \rho_{h,o}$. Economies with falling levels of oil resources should therefore invest more like "traditional" investors with cash-like liabilities.

Again, we ask ourselves which kind of investment strategies would show zero correlations with oil price movements such that we can separate investment decisions into building both a growth and a hedge portfolio. A natural candidate would be hedge fund investments. Table 4 summarizes our results. All popular hedge fund strategies we have looked at are uncorrelated to oil price movement over medium to longer time horizons. Hedge funds could therefore be an interesting add-on to investments in long government bonds.

4. Background Risk – The Impact of Resource Uncertainty

There is a vast literature on background risk,[6] *i.e.*, risk that is uncorrelated to the assets you decide upon. It exists in the background of the decision maker. Pension funds, for example, are exposed to background risk in the form of mortality risk, that is, independent of interest rate or equity risk.

How can we translate this idea into our framework for finding the optimal allocation for an SWF? To the best of our knowledge, this has not been addressed in the theoretical literature. So far we have assumed that θ (*i.e.*, the fraction of SWF assets to total sovereign wealth) is known with certainty in (5), *i.e.*, that the value of oil reserves is known to the decision maker. However, the size of an oil field is not known with great precision. In addition, government claims are sometimes legally disputed (among neighbouring countries) and new undiscovered fields might yet be found. Hence, θ might be best thought of as a random variable. We assume the fraction of financial wealth relative to total (financial and oil) wealth follows a uniform distribution around the government estimate of $\bar{\theta}$.

5 - For the more general case we get $\frac{dw_h^*}{d\theta} = \frac{-\left(\mu_h - \mu_g \frac{\rho_{h,g}\sigma_h}{\sigma_g}\right) + \lambda\rho_{h,o}\sigma_h\sigma_o}{\lambda\left(1-\rho_{g,h}^2\right)\sigma_h^2\theta^2}$. However, the intuition does not change.
6 - See Gollier (2001) for a review.

More precisely, we assume

(16) $$\tilde{\theta} \sim U(\bar{\theta} - \varepsilon, \bar{\theta} + \varepsilon)$$

It seems natural to further assume independence of the background risk on the level of available oil reserves and asset risk. The joint probability density function can then be written

(17) $$f(\theta, r_a) = f(\theta) f(r_a) = \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{r_a - \mu_a}{\sigma_a}\right)^2} \frac{1}{(\bar{\theta} + \varepsilon) - (\bar{\theta} - \varepsilon)}$$

We are looking for

(18) $$Var(\tilde{\theta}\tilde{r}_a) = E\left[(\tilde{\theta}\tilde{r}_a)^2\right] - E[\tilde{\theta}\tilde{r}_a]^2$$

in order to calculate portfolio risk. Given the joint probability density (17), this amounts to integrating over the joint probability density where

(19) $$E\left[(\tilde{\theta}\tilde{r}_a)^2\right] = \int_{-\infty}^{\infty}\int_{\bar{\theta}-\varepsilon}^{\bar{\theta}+\varepsilon} (\theta r_a)^2 f(\theta, r_a)\, d\theta\, dr_a = \frac{1}{3}\left(\varepsilon^2 + 3\bar{\theta}^2\right)\left(\mu_a^2 + \sigma_a^2\right)$$

For $\varepsilon \rightarrow 0$ we converge to the well known expression from undergraduate statistics $Var(\tilde{\theta}\tilde{r}_a) = \bar{\theta}^2 \sigma_a^2$.

What does (20) imply? As long as we have background risk in the form of uncertainty around the size of oil reserves the optimal asset allocation for the SWF (we focus on the case with uncorrelated assets and oil returns for simplicity) becomes

(21) $$w_{br}^* = \frac{1}{\lambda} \frac{\bar{\theta}\mu_a}{\bar{\theta}^2 \sigma_a^2 + \varepsilon^2 \frac{\mu_a^2 + \sigma_a^2}{3}}$$

We now compare this with the solution in absence of background risk $w^* = \frac{1}{\bar{\theta}} \frac{\mu_a}{\lambda \sigma_a^2}$ by building the quotient.

(22) $$\frac{w^*}{w_{br}^*} = 1 + \frac{\varepsilon^2 \left(\mu_a^2 + \sigma_a^2\right)}{\bar{\theta}^2 \sigma_a^2} > 1$$

which will always be greater than 1. An increase in background risk will lead to a decrease in risk taking for the SWF. The effect becomes stronger the more volatile our risky asset is. Empirically, we should observe that SWFs with larger resource uncertainty should invest less aggressively and vice-versa. We would also expect that economies with low reserves relative to financial wealth are less affected by resource uncertainty.

5. Asset Allocation and Oil Reserves over Time

What will drive the optimal asset allocation for an SWF over time? How is the SWF expected to shift its assets? How fast will the financial wealth of oil-rich countries accumulate? We might want the answer to these questions either to solve the dynamic portfolio choice problem for an SWF or to asses the change in global financial flows for the coming years.

A brief look at the myopic one-period solution in (5) reveals that the fraction of risky assets is driven by financial wealth relative to resource wealth. For a "young" SWF where financial wealth is low relative to resource wealth a more risky asset allocation is optimal, while mature SWFs with large assets relative to natural resources should dial back their risks. To decide on the optimal asset allocation over time we therefore need to calculate the optimal extraction policy, *i.e.*, how fast is oil wealth transformed into financial wealth. If expected oil price changes are high relative to asset returns (opportunity costs of keeping resources under ground) we would expect slower oil extraction and therefore a lower ratio of financial wealth to resource wealth. Also, if extraction technology improves (lower extraction costs) we expect a faster "oil-to-equity transformation".

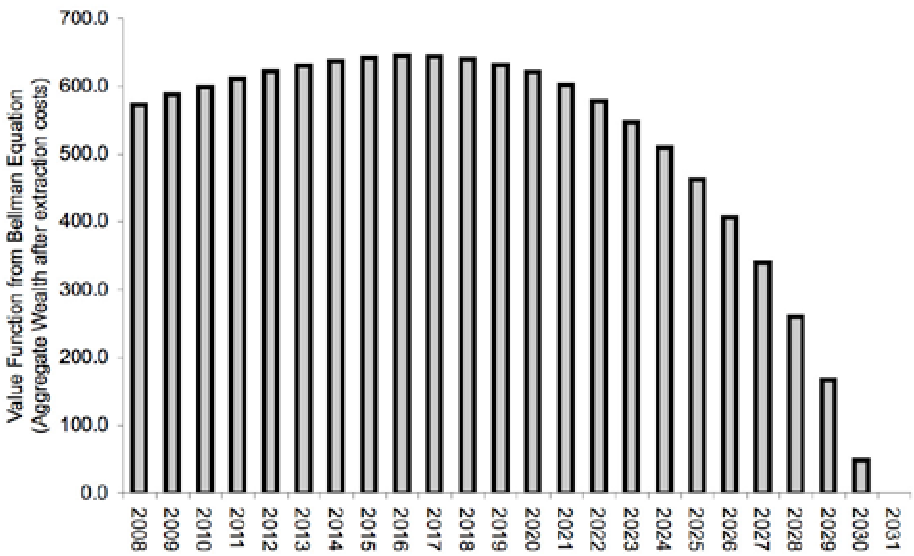
We choose a standard dynamic programming framework (with fixed time horizon) to address this question, where the optimal extraction problem is solved by recursively working backward through the well known Bellman equation, where for $t = 0,...,n-1$.

$$V_t = \max_{\xi_t}\left(f_t\xi_t - \phi\xi_t^2\right) + \tfrac{1}{1+r}V_{t+1}\left(o_t - \xi_t\right) \quad (23)$$

Here f_t represents the projected oil price for period t, ξ_t stands for the level of extraction, *i.e.*, $f_t\xi_t$ represents oil revenues, the state variable, o_t, denotes oil reserves and the cost function for oil extraction is assumed to be quadratic in extraction with a calibration parameter ϕ. In order to solve (23) we also need a terminal condition, *i.e.*, that in the last period all remaining oil wealth will be extracted (at whatever cost it takes), *i.e.*, for $t = n: \; V_{t+n} = f_t o_t - \phi o_t^2$. In other words: each period an oil extraction decision ξ_t is made that leads to a reduction in oil reserves $o_t = o_{t-1} - \xi_t$ and the present value after cost extractions is maximized. The optimal extraction policy has to counterbalance the desire to extract all oil at once to get immediate rather than very distant cash flows against the rising extraction costs of doing so.[7]

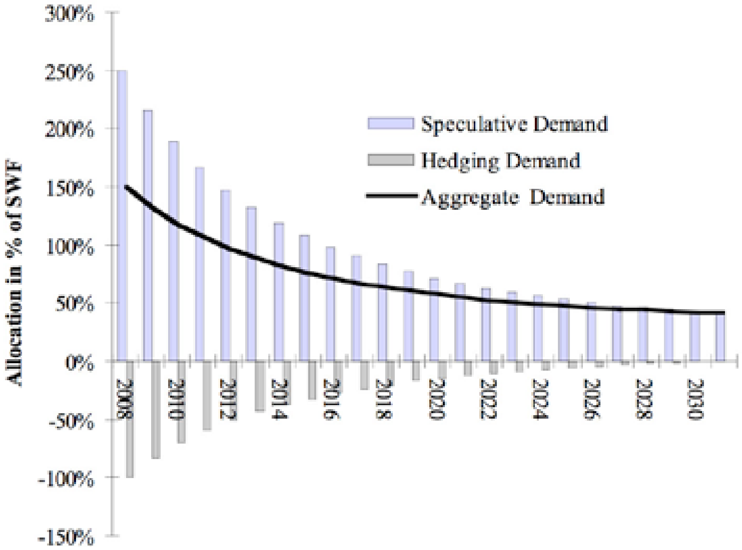
We calibrate our calculations with the following sets of assumptions. Initial oil reserves in Norway are 9,947 million barrels at a price of USD 70 a barrel. Current oil extraction is assumed to be around 720 million barrels a year. Assuming extraction is optimal (myopic) at current levels we can calibrate the cost function parameter from $\phi = \frac{f_0}{2\xi_0} = 0.00000005$. This is obviously equivalent to assuming that a (current, *i.e.*, October 2008) price of USD70 is equal to marginal production costs. Oil price growth is expected to be around 5% per annum, alongside a 5% risk-free rate, *i.e.*, $\beta = \frac{1}{1+r} = 0.9524$. We also assume the government to be capturing 100% of revenues from oil extraction.

Figure 2. Evolution of oil wealth (adjusted for extraction costs) over time. Oil wealth is directly calculated from the Bellman value function, *i.e.*, we plot V_t^* from (23) for t - $0,...,n$, where 0 represents the year 2008. This is a better measure of wealth than simply multiplying remaining oil reserves by oil prices. After all, oil needs to be extracted first.



7 - An introduction into dynamic programming is found in Bertsekas (1976). The interested reader can get code for NUOPT™ for S-PLUS™ from the author on request.

Figure 3. Optimal SWF allocation to risky assets over time. We assumed $\mu_a = 5, \sigma_a = 20, \lambda = 0.03$ as well as $\mu_o = 0, r = 5, \sigma_o = 40, \rho = 0.1$. Aggregate demand for the risky asset arises from speculative and hedging demand. Over time, hedging demand reaches zero as resources become depleted.



The result of the above calculations can be found in figure 2. It is worth noting that we do not equate oil wealth with the market value of oil reserves. In fact, we argue that oil wealth is defined by the optimal extraction policy (whose aim is to maximize oil wealth) and therefore can be found from V_0^* from (23). In our example oil wealth is assumed to be USD 572 billion instead of USD 696 billion (9,947 million barrels times current price of USD 70/barrel).

As oil wealth becomes depleted the relative importance of financial assets to natural resources shifts. We simply calculate θ_t across time (assuming a starting position of $\theta = \frac{1}{6}$, *i.e.*, financial wealth of USD 114.6 billion USD) and substitute this into (5). This allows us to estimate the evolution of risky assets (as a fraction of financial wealth) over time. The results for speculative and hedging demand are given in figure 3.

The SWF starts out as an aggressive investment vehicle with a leveraged position (150% exposure) in the risky asset. As time goes by, hedging demand is reduced but so is speculative demand. Hedging demand is negative for positively correlated assets, *i.e.*, the SWF fund scales back risks it would otherwise take on a stand-alone basis. With no resources left, the SWF would invest about 42% in the risky asset with the remaining allocation in cash.

The above framework can be easily applied to a multi-asset context. In this case it is trivial to expand (11) and (12), while the optimal extraction policy will remain the driving force for portfolio adjustments across time.

6. Conclusions

Sovereign wealth funds might be a new set of investors but classic portfolio choice still applies. We find that the SWF decision-making problem can be modelled as optimal asset allocation with endowed, non-tradable wealth. Closed-form solutions are readily available and allocations can be separated with the usual two-fund separation. The first fund being an optimal growth portfolio with the second an oil price risk hedging portfolio. We also investigate the impact of resource uncertainty on optimal asset allocation. An SWF of a sovereign entity with considerable resource uncertainty might find it optimal to invest less aggressively than an SWF with well established oil resources. Finally, we showed how we can model optimal asset allocation over time as a function of the optimal oil extraction policy. Maturing SWF funds will invest less aggressively, while recently funded SWFs need to be run very aggressively to diversify total wealth.

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