

The Benefits of Structured Products in Asset-Liability Management

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Abstract

This paper introduces a continuous-time dynamic asset allocation model for an investor facing liability constraints in the presence of inflation and interest rate risks. When funding ratio constraints are explicitly accounted for, the optimal policies, for which we obtain analytical expressions, are shown to extend standard Option-Based Portfolio Insurance (OBPI) strategies to a relative risk context, with the liability-hedging portfolio replacing the risk-free asset. We also show that the introduction of maximum funding ratio targets would allow pension funds to decrease the cost of downside liability risk protection while giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero.

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1 Introduction

Market difficulties at the turn of the millennium have drawn attention to risk management practices of institutional investors in general and defined benefit pension plans in particular. What has been labeled as a “perfect storm of adverse market conditions” has devastated many corporate pension plans, with negative equity market returns that have eroded plan assets at the same time as declining interest rates have increased market-to-market value of benefit obligations. In 2003, the defined benefit pension plans for the companies included in the S&P 500 and the FTSE 100 index faced a cumulative deficit estimated at \$225 billion and £55 billion, respectively, while the worldwide deficit reached an estimated \$1,500 to \$2,000 billion.¹ To better understand the magnitude of the crisis, and the scale and rapidity of the deterioration in pension funding status, it is perhaps worth noting that in the United States, defined benefits plans from S&P500 companies were enjoying a total *surplus* of \$239 billion at the end of 1999, a mere 3 years earlier!² While it is too early to assess the exact impact of the current credit crisis on pension fund financial health, a recent report has provided evidence that deficits are again widening for European pension plans.³ In October 2008, MSCI Europe pension plan assets have fallen by an estimated 19% on average since the start of the year, due to the fall in equities and other assets, while obligations have fallen only by 10% year to date based on a 97bps rise in the AA bond rates.

That institutional investors have been so dramatically affected by market downturns is perhaps surprising given that an increasingly thorough range of investment techniques have been developed over the past few years, which allow investors to tailor the risk profile of their portfolios in a more efficient way. These techniques, which allow their user to enjoy a non-linear (typically convex) exposure with respect to the return on traditional asset classes, can be conveniently accessed under the form of a variety of packaged products (known as portfolio insurance products) manufactured by investment banks. From an academic standpoint, it has early been recognized that structured products are natural investment vehicles for institutional investors, who have a particularly strong preference for non-linear payoffs because of the non-linear nature of the liability constraints they face (see for example Draper and Shimko (1993)). Leland (1980) has shown in a fairly general context that investors whose risk tolerance increases with wealth more rapidly than the average will rationally wish to obtain portfolio insurance. This is the case in particular for institutional investors whose portfolio value must at all cost exceed a given value, but thereafter can accept reasonable risks. From a pragmatic standpoint, taking for example the case of pension funds, it is clear that a small change in the probability of

¹See Standard Life Investments (2003) and Watson Wyatt (2003).

²See Standard Life Investments (2003).

³“Valuation & Accounting No Respite from Pension Risk”, Morgan Stanley Research Europe, October 2008.

extreme contribution rates is typically considered much more important than an equal change in the probability of an extremely high refund.

In fact, previous research (see in particular van Capelleveen et al. (2004)) has shown that introducing suitably designed option portfolios could add significant value in pension fund management. In a nutshell, structured products with convex payoffs allow investors to profit from the equity risk premium, which is often needed to match the returns on liability-driven benchmarks, without being fully exposed to the downside risk associated with investing in stocks. Formalizing this intuition, Goltz et al. (2008) show that structured products with convex payoffs are particularly appealing for investors with a focus on downside risk protection. One key limitation of the existing research on the subject, which has provided useful insights into the benefits of option-based portfolio strategies, is that it is of a strictly positive nature and focuses on heuristic strategies based on plain vanilla option products, with no analysis of how such strategies compare to optimal strategies given investors' preferences, and also funding ratio constraints that have been introduced by the regulator in most developed countries in an attempt to provide direct incentives for pension funds to increase the focus on risk management, and in view of protecting the interests of beneficiaries.

From an academic perspective, a number of papers have focused on extending Merton's intertemporal selection analysis (see Merton (1971)) to account for the presence of liability constraints in the asset allocation policy. Broadly speaking, asset-liability management (ALM) distinguishes itself from pure asset management by the fact that what matters from an ALM perspective is not total terminal wealth, but how terminal wealth compares to terminal value of future liabilities. More precisely, the presence of future liability commitments is accounted for by a focus on terminal wealth net of the value at horizon of future liability payments, a quantity known as the pension fund *surplus* (or as the pension fund *deficit* when it is negative).⁴ A first step towards the application of optimal portfolio selection theory to the problem of pension funds has been taken by Merton (1993) himself, who studies the allocation decision of a university that manages an endowment fund. In a similar spirit, Rudolf and Ziemba (2004) have formulated a continuous-time dynamic programming model of pension fund management in the presence of a time-varying opportunity set, where state variables are interpreted as currency rates that affect the value of the pension's asset portfolio. Also related is a paper by Sundaresan and Zapatero (1996), which is specifically aimed at asset allocation and retirement decisions in the case of a pension fund. van Binsbergen and Brandt (2007) complement this early work by analyzing how various regulatory rules with respect to the valuation of liabilities impact optimal investment decisions, while Detemple and Rindisbacher

⁴In fact, defined benefit pension commitments represent a short position in collateralized defaultable bonds issued by the sponsor company and privately held by employees, where the assets of the pension plans are the collateral, in exchange for which the company receives the present value of lower wage demands.

(2008) consider an ALM problem with a limited tolerance for a shortfall in the funding ratio at the terminal date. In a nutshell, the main insight from this strand of the literature is that the presence of liability risk induces the introduction of a specific hedging demand component in the optimal allocation strategy, as typical in intertemporal allocation decisions in the presence of stochastic state variables. On the other hand, relatively little is known on how the presence of formal funding ratio constraints impacts the optimal asset allocation policy.

This paper precisely extends the afore-mentioned literature on asset allocation decisions with liability commitments by analyzing the impact of formal funding ratio constraints in the context of a continuous-time model for intertemporal allocation decisions. Given that interest rate and inflation uncertainty are the two main risk factors impacting pension liability values, we cast the problem in a setting with stochastic interest and inflation rates. Using the martingale approach to portfolio optimization problems, we obtain analytical solutions that allow us to confirm that the optimal strategy involves a fund separation theorem that legitimates investing in a liability-hedging portfolio, in addition to the standard performance-seeking portfolio (speculative demand). Recognizing that the presence of minimum funding ratio constraints, whether desirable or not, should affect the optimal allocation policy, we then provide the formal solution to the asset allocation problem in the presence of such constraints. When funding ratio constraints are explicitly accounted for, the optimal policies, for which we obtain analytical expressions, are shown to be reminiscent of Option-Based Portfolio Insurance (OBPI) strategies, which they extend to a relative (with respect to liabilities) risk context. We also show that the introduction of maximum funding ratio targets would allow pension funds to decrease the cost of downside liability risk protection while giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero.

The rest of the paper is organized as follows. In Section 2, we introduce a formal continuous-time model of dynamic asset allocations decisions in the presence of liability commitments. In Sections 3 and 4, we analyze the impact of formal minimum and/or maximum funding ratio constraints on optimal allocation strategies. In Section 5, we provide a numerical illustration. In Section 6, we present a conclusion as well as suggestions for further research. Technical details and proofs of the main results are relegated to a dedicated Appendix.

2 A Formal Model of Asset-Liability Management

In this section, we introduce a general continuous-time asset allocation model for a pension fund facing liability constraints. This continuous-time stochastic control approach to asset-liability management (ALM) is appealing in spite of its highly stylized nature because it leads to tractable solutions, allowing one to explicitly understand the various mechanisms affecting

the optimal allocation strategy.

We let $[0, T_0]$ denote the (finite) time span of the economy, where uncertainty is described through a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that financial markets are frictionless, and we denote by r_t the nominal short-term interest rate at time t .

2.1 State Variables

Regarding the liability side, inflation risk and interest rate risk appear as the two most relevant risk factors. This is because pension benefits are typically indexed with respect to inflation, and on the other hand the typically long duration of liability payments make their current value highly sensitive to changes in interest rates. As a result, it is critically needed for our model to incorporate both stochastic interest and inflation rates. In what follows, we model the nominal short-term interest rate as an Ornstein-Uhlenbeck process (see Vasicek (1977)):

$$dr_t = a(\bar{b} - r_t) dt + \bar{\sigma}_r d\bar{z}_t^r \quad (2.1)$$

where \bar{z}^r follows a standard Wiener process under \mathbb{P} .⁵ On the asset side, we assume that the menu of asset classes includes a unit zero-coupon bond with payoff 1 at maturity τ_1 , where the maturity $\tau_1 \in [0, T_0]$ and will be specified later, and a price $B(t, \tau_1)$ at time t given by a deterministic function of (t, r_t, τ_1) which is given in Appendix A.

Our model also accounts for stochastic inflation, by assuming that the price index follows a Geometric Brownian motion:

$$\frac{d\Phi_t}{\Phi_t} = \bar{\varphi} dt + \bar{\sigma}_\Phi d\bar{z}_t^\Phi \quad (2.2)$$

where \bar{z}^Φ is a \mathbb{P} -Brownian motion correlated to \bar{z}^r . $\bar{\varphi}$ represents the instantaneous expected inflation rate, which we assume to be constant for simplicity.⁶ So as to stay within a complete market environment, we assume on the asset side that there exists an inflation-indexed unit zero-coupon bond of maturity τ_2 , i.e., a bond with payoff given by Φ_{τ_2} . The price at time t of the inflation-linked bond is denoted by $I(t, \tau_2)$, a deterministic function of (t, r_t, Φ_t, τ_2) which is also given in Appendix A.

Moreover, we assume that there exists one stock index with value S_t at time t , where S evolves as:

$$dS_t = S_t[\mu_{S,t} dt + \sigma_{S,t} dz_t^S] \quad (2.3)$$

where z^S is a \mathbb{P} -Brownian motion correlated to \bar{z}^r and \bar{z}^Φ . Because the number of stocks is identical to the number of risk factors impacting stock returns, the stock market is complete, as

⁵It is well-known that a limitation of this model is that it allows for negative interest rates with positive probability, although this probability will be small for a proper choice of parameter values.

⁶Brennan and Xia (2002) and Munk et al. (2004) assume that the expected inflation rate follows an Ornstein-Uhlenbeck process.

was the bond market. In this complete market setting, assuming away arbitrage opportunities implies that there exists a unique market price of risk vector $\boldsymbol{\lambda}$, which we will assume to be constant, and a unique equivalent probability measure \mathbb{Q} under which discounted prices are martingales.

(2.1), (2.2) and (2.3) can be rewritten using a $n+2$ -dimensional Brownian motion \mathbf{z} under \mathbb{Q} (see e.g. Shreve (2005)):

$$\begin{aligned} dr_t &= a(b - r_t) dt + \boldsymbol{\sigma}'_r d\mathbf{z}_t \\ \frac{d\Phi_t}{\Phi_t} &= \varphi dt + \boldsymbol{\sigma}'_\Phi d\mathbf{z}_t \\ dS_t &= S_t[r_t dt + \boldsymbol{\sigma}'_S d\mathbf{z}_t] \end{aligned}$$

where $b = \bar{b} - \boldsymbol{\sigma}'_r \boldsymbol{\lambda} / a$ and $\varphi = \bar{\varphi} - \boldsymbol{\sigma}'_\Phi \boldsymbol{\lambda}$ are respectively the long-term mean of r and the drift of Φ under the new probability measure \mathbb{Q} . $\boldsymbol{\sigma}_r$, $\boldsymbol{\sigma}_\Phi$ and $\boldsymbol{\sigma}_S$ are the volatility vectors of r , Φ and S . In Appendix A, we derive the expressions for the volatility vectors of both types of bonds, $\boldsymbol{\sigma}_B(\cdot, \tau_1)$ and $\boldsymbol{\sigma}_I(\cdot, \tau_2)$. We denote with $\boldsymbol{\sigma}_t$ the square matrix with columns $\boldsymbol{\sigma}_S$, $\boldsymbol{\sigma}_B(\cdot, \tau_1)$ and $\boldsymbol{\sigma}_I(\cdot, \tau_2)$.

2.2 Liability and Net Wealth of the Pension Fund

We consider a pension fund managing financial assets and paying a stream of pension payments. For simplicity, we do not model the stream of contributions from the sponsor company, and assume instead that it can be summarized by an initial endowment A_0 to the pension fund.⁷ This initial wealth can be invested in the n stocks, the two zero-coupon bonds, and a locally risk-free asset, whose value S^0 is the continuously compounded risk-free nominal interest rate. We denote with $\boldsymbol{\omega}_t$ the following vector of weights at time t : weight of the stock index, weight of the nominal bond and weight of the indexed bond. We denote by A_t the asset value of the pension fund at time t , which is given as the market value of its asset portfolio after taking into account pension payments *before* t . Hence, the change in A over a small period $[t, t + dt]$ is equal to the return on the financial portfolio, from which is subtracted the pension payment over the period. Mathematically, this is written as:

$$dA_t = A_t[r_t + \boldsymbol{\omega}'_t \boldsymbol{\sigma}'_t \boldsymbol{\lambda}] dt + A_t \boldsymbol{\omega}'_t \boldsymbol{\sigma}'_t d\mathbf{z}_t - dV_t \quad (2.4)$$

where V is an increasing right-continuous stochastic process with left limits, so that dV_t represents the pension payment over the interval $[t, t + dt]$. This model of pension payments is

⁷In a more general setting, one should allow for an endogenous contribution policy from a sponsor company perspective. This would however require a formal dynamic capital structure model in the presence of a pension funds (see suggestions for further research in the conclusion).

inspired by models of lump-sum dividend payments for stocks (see for example Section 6.M of Duffie (2001)). In fact, this general formulation can encompass various situations with either continuous or discrete pension payments. Hence we can assume that $dV_t = l_t dt$, where l is an \mathcal{F} -adapted process, a model that captures a stream of continuous payments. Alternatively, we can take $dV_t = \sum_{i=1}^n l_{t_i} dH_t^{t_i}$, where $0 \leq t_1 < \dots < t_n = T_0$. Here H^{t_i} is the Heaviside function, $H_t^{t_i} = \mathbb{1}_{\{t \geq t_i\}}$, l_{t_i} is an \mathcal{F}_{t_i} -measurable random variable. This model captures a stream of discrete payments from date 0 to date T_0 . In what follows, we focus on a specific case of the discrete payment model, where we allow for a single payment at date T_0 , a situation we shall refer to as the *zero-coupon case*.

From (2.4), an important equation can be derived, which we will refer to as the *budgetary constraint*. It explains how the net wealth at any date t can be computed from the net wealth at a later date s and the intermediate pension payments. In the zero-coupon case, we have, for $t \leq s < T_0$:

$$A_t = \mathbb{E}_t \left[\frac{M_s}{M_t} A_s \right] \quad (2.5)$$

This property follows from Proposition 2.2 in Cox and Huang (1988).

Since the financial market is complete, the stream of future payments can be valued as the dividend flow of a financial asset. Hence the quantity

$$L_t = \mathbb{E}_t^{\mathbb{Q}} \left[\int_{]t, T_0]} e^{-\int_t^s r_u du} dV_s \right], \quad t \leq T_0 \quad (2.6)$$

is the price that an agent would have to pay at time t to receive the payment stream dV from date t excluded to date T_0 included.⁸ We will take throughout the paper $l_t = n_t \Phi_t$, where n is a *nonnegative* deterministic function of time representing the size of the population to which benefits will be provided for. Hence, the pension fund is pre-committed to pay n_t in real terms at time t , which amounts to a nominal payment of $n_t \Phi_t$. In the zero-coupon case, a single payment takes place at time T_0 , so (2.6) can be simplified into:

$$L_t = n_{T_0} I(t, T_0) \quad (2.7)$$

In this particular case, the volatility vector of L is $\sigma_I(\cdot, T_0)$. In the theoretical results of this paper, we restrict the problem of optimal allocation decisions to this zero-coupon case, that is we will assume that L_t is given by (2.7). This assumption does not really simplify the computations, but it leads to allocation policies that are more easily interpreted. The general case, in which several payments take place, is addressed in Martellini and Milhau (2008) (referred to as MM08 from now on).⁹

⁸In this paper, we assume away the presence of default risk on liability payments. See the Conclusion for a discussion of this important aspect.

⁹The base case empirical exercises of Section 5, however, will assume that several payments take place before date T_0 .

2.3 Objectives and Optimal Asset Allocation Decisions

In terms of objectives, it is customary to assume that the preferences of the investor are captured by an expected utility framework:

$$\max_{\omega} \mathbb{E}[u(A_T)] \quad (2.8)$$

where u is a growing and concave utility function and T is some horizon date lying between 0 (the initial date) and T_0 (the date of the latest pension payment, beyond which the pension fund has no longer any reason to exist). In what follows we take u to be the constant relative risk aversion (CRRA) utility function, defined as:

$$\begin{aligned} u(x) &= \frac{x^{1-\gamma}}{1-\gamma} & \text{for } x > 0 \\ u(x) &= -\infty & \text{for } x \leq 0 \end{aligned}$$

where γ lies in $[1, \infty[$. If $\gamma = 1$ we obtain the logarithmic utility function.

One key problem with this objective in an asset-liability management context is that it fails at recognizing that further liability payments are scheduled beyond the horizon T for any choice of $T < T_0$. Hence, while past liability payments have been accounted for in the dynamics of asset value, the presence of future liability payments will not be incorporated in the pension fund objective. One natural approach to tackle this problem consists of recognizing that terminal wealth at date T is made of a long position in the asset portfolio with value A_T , but also involves a short position in the liability portfolio with value L_T . Hence, preferences of the pension fund manager are expressed over the terminal surplus, as opposed to over the terminal asset value, so that the optimization program reads:

$$\max_{\omega} \mathbb{E}[u(A_T - L_T)]$$

Another related approach to account for the presence of liability payments beyond the horizon consists of introducing an additional state variable, the funding ratio, defined as the ratio of assets to liabilities:

$$F_t \equiv \frac{A_t}{L_t} \quad (2.9)$$

which is well-defined as long as L_t is not zero. This quantity is commonly used in industry practice, where a pension trust is said to be overfunded when the funding ratio is greater than 100%, to be fully funded when the funding ratio equals 100%, and to be underfunded when the funding ratio is lower than 100%. One may therefore capture the pension fund objective as:

$$\max_{\omega} \mathbb{E}[u(F_T)] \quad (2.10)$$

if pension payments are not zero after the horizon T . This objective, which is perhaps the most natural since it recognizes that what really matters in pension fund management is not the value of the assets per se, but how asset value compares to liability value at each point in time, has been used for example by van Binsbergen and Brandt (2007). Maximization of expected utility of the funding ratio accounts for the presence of future liability payments since by definition (see (2.9)) it is defined as the asset value net of past liability payment expressed in terms of number of units of the current value of future liability payments. From an interpretation standpoint, this amounts to using the liability value process $(L_t)_{t \geq 0}$, as opposed to the bank account, as the numeraire, a natural choice given that what matters for the pension fund is not asset value per se, but how asset value compares to liability value. If liability commitments consist of a single inflation indexed-payment, then this objective is equivalent to maximizing real wealth, as opposed to nominal wealth, a problem considered Brennan and Xia (2002). We choose in what follows to present detailed results for the case of the funding ratio, which amounts to using the current value of all future liability payments as the natural numeraire portfolio.¹⁰

To obtain the solution to the previous program, we use the martingale approach in complete markets developed by Karatzas et al. (1987) and Cox and Huang (1989), which involves two main steps. In a first step, the dynamic program is re-written in a static form that allows for the derivation of the optimal terminal net wealth, A_T^* . In general, this can be done in the general case, as well as in the zero-coupon case. Subsequently, the optimal net wealth at time t , A_t^* , is obtained by an application of (2.5) with $s = T$. To obtain an explicit expression of A_t^* , we need in general make the simplifying assumption that no payments take place before T . In a second step, the optimal dynamic portfolio strategy is obtained as the replicating portfolio for the optimal terminal wealth. In fact, applying Ito's lemma to the process A^* and identifying the diffusion terms with (2.4) yields the optimal portfolio strategy. Details of derivation are relegated to appendices. The following proposition presents the expression for the optimal policy and optimal asset value process for (2.10) in the zero-coupon case.¹¹

Proposition 1 *The solution to (2.10) in the zero-coupon case is given by the following asset strategy:*

$$\omega_t^{*u} = \alpha(T_0 - t) \left(1 - \frac{1}{\gamma}\right) \sigma_t^{-1} \sigma_r + \left(1 - \frac{1}{\gamma}\right) \sigma_t^{-1} \sigma_\Phi + \frac{1}{\gamma} \sigma_t^{-1} \lambda$$

¹⁰Solving the optimal allocation problem when preferences are expressed in terms of the surplus $A_T - L_T$ or in terms of the funding ratio $\frac{A_T}{L_T}$ generates in fact, and quite unsurprisingly, very similar results. In particular, we obtain the exact same expression for the relationship between the optimal wealth conditional upon solving the unconstrained problem and the optimal wealth conditional upon solving the constrained problem when using either the surplus or funding ratio as objectives. These results can be found in MM08.

¹¹The general case is addressed in MM08.

which can also be written as:

$$\omega_t^{*u} = \frac{e'_{n+2}\sigma_t^{-1}\lambda}{\gamma}\omega_t^{PSP} + [\alpha(T_0 - t)e'_{n+2}\sigma_t^{-1}\sigma_r + e'_{n+2}\sigma_t^{-1}\sigma_\Phi] \left(1 - \frac{1}{\gamma}\right)\omega_t^{LMP}(T_0)$$

where:

$$\omega_t^{PSP} \equiv \frac{\sigma_t^{-1}\lambda}{e'_{n+2}\sigma_t^{-1}\lambda}, \quad \omega_t^{LMP}(T_0) = \frac{\sigma_t^{-1}\sigma_I(t, T_0)}{e'_{n+2}\sigma_t^{-1}\sigma_I(t, T_0)}$$

Proof. This proposition is obtained as a particular case of Proposition 2 for $k = 0$. ■

We find that the solution involves the standard performance-seeking portfolio (PSP) and a liability-hedging or liability-matching portfolio (LMP). This portfolio has the following property, which is typical of intertemporal hedging demand terms in dynamic asset allocation models (see Merton (1973)): $\omega_t^{LMP}(T_0)$ maximizes the correlation between the returns on the asset portfolio and the return on the present value of future pension payment. In fact, in this complete market setting, the maximum correlation achieved is equal to 1. In case the maturity of the inflation-linked bond coincides with the date of the unique payment T_0 , the liability-matching portfolio is fully invested in this inflation-linked bond; otherwise it involves the combination of cash and the inflation-linked-bond needed for reaching the target duration. It should be noted that the optimal portfolio strategy does not involve a *separate* interest rate hedging component. While interest rate risk impacts the asset value, it also impacts liability value in such a way that the net impact at the funding ratio level is trivial.

3 Introducing Funding Ratio Constraints

As discussed before, funding ratio constraints, whether desirable or not, are dominant in pension funds' environment. The allocation strategy presented in Proposition 1 is in fact not optimal in the presence of liability constraints. As opposed to following an unconstrained strategy, and eventually requesting additional contributions, we now turn to the analysis of the optimal allocation strategy when funding ratio constraints are explicitly accounted for. To this end, we now consider the following optimization program with explicit constraints:

$$\max_{\omega} \mathbb{E}[u(F_T)] \tag{3.1}$$

such that $A_T \geq kL_T$ almost surely. Imposing an explicit lower bound intuitively means that the pension fund has infinitely low utility from funding ratios below k . This leads us to consider the following equivalent program:

$$\max_{\omega} \mathbb{E}[\tilde{u}^k(F_T)] \tag{3.2}$$

where \tilde{u}^k is a (non-smooth) utility function defined as follows:¹²

$$\begin{cases} \tilde{u}^k(x) = u(x) & \text{if } x \geq k \\ \tilde{u}^k(x) = -\infty & \text{if } x < k \end{cases} \quad (3.3)$$

It should be noted that because of the budget constraint (2.5), we need to have $A_0 \geq kL_0$ in order to get $F_T \geq k$ almost surely. Note also that the complete market assumption is critical here, since the presence of a non hedgeable source of risk would make it impossible for the constraint to hold almost surely.

3.1 The Constrained Solution

We now solve the optimization program when the minimum funding ratio is explicitly introduced in the investor's objective. So as to better analyze the results, we focus on the zero-coupon case, where we can have an explicit expression for the optimal dynamic allocation strategy. In the general case (see MM08), where payments take place before T , we would only have semi-explicit expressions for the optimal strategy. But the optimal terminal net wealth would be the same as here.

Proposition 2 *The solution to (3.1) in the zero-coupon case is given by:*

$$\begin{aligned} \omega_t^{*k} = & \alpha(T_0 - t) \left[1 - \frac{1}{\gamma} \left(1 - \frac{k\mathcal{N}(-d_{2,t})L_t}{A_t^{*k}} \right) \right] \sigma_t^{-1} \sigma_r \\ & + \left[1 - \frac{1}{\gamma} \left(1 - \frac{k\mathcal{N}(-d_{2,t})L_t}{A_t^{*k}} \right) \right] \sigma_t^{-1} \sigma_\Phi + \frac{1}{\gamma} \left(1 - \frac{k\mathcal{N}(-d_{2,t})L_t}{A_t^{*k}} \right) \sigma_t^{-1} \lambda \end{aligned}$$

where:

$$\begin{aligned} d_{1,t} &= \frac{1}{\frac{1}{\gamma} \sqrt{\int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds}} \left[\ln \frac{\xi A_t^{*u}}{kL_t} + \frac{1}{2\gamma^2} \int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds \right] \\ d_{2,t} &= d_{1,t} - \frac{1}{\gamma} \sqrt{\int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds} \end{aligned}$$

and where the optimal net wealth process is given by:

$$\begin{aligned} A_t^{*k} &= kL_t + \mathcal{N}(d_{1,t})\xi A_t^{*u} - kL_t \mathcal{N}(d_{2,t}) \\ A_T^{*k} &= kL_T + (\xi A_T^{*u} - kL_T)^+ \end{aligned}$$

¹²Marginal utility exhibits a discontinuity at k , a feature that prevents from applying the traditional martingale approach of Cox and Huang (1989). Such portfolio selection problems with nonsmooth preferences have been studied e.g. in Schachermayer (2001) and Bouchard et al. (2004) in a more general setting, where the financial market is not assumed to be complete.

while constant ξ is chosen so that the budget constraint $A_0^* = A_0$ holds.

The optimal portfolio can also be expressed as:

$$\begin{aligned} \omega_t^{*k} = & [\alpha(T_0 - t)e'_{n+2}\sigma_t^{-1}\sigma_r + e'_{n+2}\sigma_t^{-1}\sigma_\Phi] \left[1 - \frac{1}{\gamma} \left(1 - \frac{k\mathcal{N}(-d_{2,t})L_t}{A_t^{*k}} \right) \right] \omega_t^{LMP}(T_0) \\ & + \frac{e'_{n+2}\sigma_t^{-1}\lambda}{\gamma} \left(1 - \frac{k\mathcal{N}(-d_{2,t})L_t}{A_t^{*k}} \right) \omega_t^{PSP} \end{aligned}$$

Proof. See Appendix B. ■

The expression for the optimal investment strategy and wealth process are strongly reminiscent of OBPI (Option-Based Portfolio Insurance) strategies, which the present setup extends to an asset-liability relative risk management context. Hence, the terminal payoff A_T^{*k} is dynamically replicated by the following strategy: at time t , the investor purchases $\mathcal{N}(d_{1,t})$ shares of the portfolio that delivers ξA_T^{*u} at time T , and $k\mathcal{N}(-d_{2,t})$ shares of the liability-matching portfolio delivering L_T .

Note, of course, that when k is zero, we recover the unconstrained optimal terminal net wealth A_T^{*u} . A comparison between the optimal terminal wealth under the unconstrained strategy and the constrained strategy can be found in the following proposition, which shows that conditional upon the funding ratio being greater than the lower bound k , the terminal wealth under the constrained strategy is lower than the terminal wealth under the unconstrained strategy. This result formalizes the intuition according to which insurance of downside risk (relative to liabilities) has a cost in terms of performance potential.

Proposition 3 *For the states of the world ω such that $F_T^{*k}(\omega) \equiv \frac{A_T^{*k}}{L_T}(\omega) > k$, or equivalently such that $[\xi A_T^{*u}(\omega) - kL_T(\omega)]^+ > 0$, we have that $A_T^{*k}(\omega) < A_T^{*u}(\omega)$.*

Proof. To see this, first note that for $k > 0$, the price of the exchange option lies between the no-arbitrage bounds given on the one hand by the intrinsic value of the option $(\xi A_0 - kL_0)^+$ and on the other hand by the underlying price ξA_0 :

$$[\xi A_0 - kL_0]^+ < Ex(0, r_0, \Phi_0, \xi A_0, kL_0) = A_0 - kL_0 < \xi A_0$$

which implies that $\xi < 1$ and also that $\xi > 1 - \frac{kL_0}{A_0}$. Of course, when $k = 0$, we have that $\xi = 1$. ■

4 Introducing (Minimum and) Maximum Funding Ratio Constraints

Given the complexity of surplus sharing rules, it is unclear whether pension funds have any utility over exceedingly large surpluses. This is emphasized by Pugh (2003), who notes: “In

practical terms, funding shortfalls are the employer’s problem and funding excesses belong to the members. (...) Given that it is impossible to return any funding excess to the plan sponsor, overfunding is not a sound philosophy for most employers. (...)”. It should also be noted that in some regulatory environments, maximum funding constraints are imposed by tax authorities to prevent the deliberate or accidental build-up of excessive assets within the pension fund. More generally, maximum funding ratios, when they are not a constraint, can be a target. In the Netherlands for example, there exists a target funding ratio that is a function of investment risk, and reaches 130% on average. In this context, and given that pension funds have preferences that likely reach satiation beyond a given funding ratio level, it seems reasonable to try and analyze how the introduction of maximum funding ratio targets would impact the optimal strategy. The idea is that by giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero, the investor can decrease the cost of downside protection.

4.1 The Solution with Upper and Lower Constraints

To this end, we consider the simultaneous introduction of a minimum and a maximum funding ratio constraints, so that the optimization program is given by (2.10) subject to the additional constraints $F_T \geq k$ and $F_T \leq k'$:¹³

$$\max_{\omega} \mathbb{E}[u(F_T)] \quad \text{s.t. } k \leq F_T \leq k' \quad (4.1)$$

In order to have the constraint $k \leq F_T \leq k'$ satisfied almost surely, the initial asset A_0 should satisfy:

$$kL_0 \leq A_0 \leq k'L_0 \quad (4.2)$$

A related program reads:

$$\max_{\omega} \mathbb{E} \left[\tilde{u}^{k,k'}(F_T) \right] \quad (4.3)$$

where \tilde{u} is a non-smooth utility function defined as follows:

$$\begin{cases} \tilde{u}^{k,k'}(x) = u(x) & \text{for } k \leq x \leq k' \\ \tilde{u}^{k,k'}(x) = -\infty & \text{for } x < k \\ \tilde{u}^{k,k'}(x) = u(k') & \text{for } x > k' \end{cases} \quad (4.4)$$

In Appendix D, we present an argument suggesting that the solution to (4.3) is identical to the solution to (4.1). In other words, this argument indicates that in program (4.3), one

¹³Introducing implicit minimum and maximum funding ratio constraints could also be done, and would require the introduction of preferences defined in piece-wise manner: $v(x) = u(x - k)$ for $x \leq \tau$ and $v(x) = u(k' - x)$ for $x \geq \tau$, where τ is some threshold value, which should for example be taken equal to $\frac{k+k'}{2}$ if one wishes to obtain a continuous utility function.

optimally implements a strategy that will not lead to funding ratios above k' , even though this is not strictly forbidden as in program (4.1). Intuitively, this is because asset allocation decisions leading to funding ratios beyond k' involve an additional risk while they do not involve any marginal utility gain.

The following proposition presents the solution to these optimization programs in the presence of minimum and maximum funding ratio constraints (or targets). As before, we focus on the zero-coupon case in order to get an explicit expression for ω_t^* . The general case, where the optimal terminal net wealth is the same as here, is addressed in MM08.

Proposition 4 *The solution to the program (2.10) with constraint $k \leq F_T \leq k'$ in the zero-coupon case is given by:*

$$\begin{aligned} \omega_t^{*k,k'} = & \alpha(T_0 - t) \left[1 - \frac{1}{\gamma} \left(1 - [k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})] \frac{L_t}{A_t^{*k,k'}} \right) \right] \sigma_t^{-1} \sigma_r \\ & + \left[1 - \frac{1}{\gamma} \left(1 - [k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})] \frac{L_t}{A_t^{*k,k'}} \right) \right] \sigma_t^{-1} \sigma_\Phi \\ & + \frac{1}{\gamma} \left(1 - [k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})] \frac{L_t}{A_t^{*k,k'}} \right) \sigma_t^{-1} \lambda \end{aligned}$$

where:

$$\begin{aligned} d'_{1,t} = & \frac{1}{\frac{1}{\gamma} \sqrt{\int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds}} \left[\ln \frac{\xi' A_t^{*u}}{k' L_t} + \frac{1}{2\gamma^2} \int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds \right] \\ d'_{2,t} = & d'_{1,t} - \frac{1}{\gamma} \sqrt{\int_t^T \|\sigma_I(s, T_0) - \lambda\|^2 ds} \end{aligned}$$

The optimal net wealth process is given by:

$$\begin{aligned} A_t^{*k,k'} = & kL_t \mathcal{N}(-d_{2,t}) + k'L_t \mathcal{N}(d'_{2,t}) + \mathcal{N}(d_{1,t}) \xi' A_t^{*u} - \mathcal{N}(d'_{1,t}) \xi' A_t^{*u} \\ A_T^{*k,k'} = & kL_T + (\xi' A_T^{*u} - kL_T)^+ - (\xi' A_T^{*u} - k'L_T)^+ \end{aligned}$$

The constant ξ' is adjusted to make the budget constraint $A_0^{*k,k'} = A_0$ hold.

Proof. We explain how to derive the optimal terminal net wealth. The first-order optimality condition for the optimization program reads:

$$\frac{1}{L_T} \left(\frac{A_T^{*k,k'}}{L_T} \right)^{-\gamma} - \nu_1 M_T + \frac{\nu_2}{L_T} - \frac{\nu_3}{L_T} = 0$$

where $kL_T \leq A_T^* \leq k'L_T$, ν_2 and ν_3 are nonnegative and $\nu_2(A_T^{*k,k'} - kL_T) = \nu_3(A_T^{*k,k'} - k'L_T) = 0$. This implies that:

$$\frac{A_T^{*k,k'}}{L_T} = k + \left[(\nu_1 M_T L_T)^{-\frac{1}{\gamma}} - k \right]^+ - \left[(\nu_1 M_T L_T)^{-\frac{1}{\gamma}} - k' \right]^+$$

The end of the proof involves the pricing of two exchange options and application of Ito's lemma. It is very similar to the proof of the single-constrained case, which is presented in Appendix B. ■

Comparing this solution to the case of the solution with lower constraint only, it can be seen that the imposition of an explicit upper bound involves a reduction of the risk budget. Intuitively, the idea is to reduce the cost of downside protection by giving up some access to the upside potential beyond the funding ratio threshold k' . This reduction of the cost of downside protection obtained through the introduction of the upper bound on the funding ratio implies that the optimal terminal asset value conditional upon the funding ratio lying between k and k' is higher when both constraints are imposed compared to the case when the lower constraint only is imposed. This is the content of the following proposition.

Proposition 5 *Let $F_T^{*,k,k'} \equiv \frac{A_T^{*,k,k'}}{L_T}$ denote the optimal terminal funding ratio when the lower bound k and the upper bound k' are imposed. Similarly, let $F_T^{*,k} \equiv \frac{A_T^{*,k}}{L_T}$ denote the optimal terminal funding ratio when only the lower bound k is imposed. For the states of the world ω such that $k < F_T^{*,k,k'}(\omega) < k'$, we have that $A_T^{*,k}(\omega) < A_T^{*,k,k'}(\omega)$.*

Proof. See Appendix C. ■

5 Empirical Testing of the Strategies

We now turn to an empirical testing of the optimal strategies discussed in the previous Section. To this end, we use a schedule of liability payments provided for by a Dutch pension fund, which name shall not be revealed. The stream of payments is displayed in Table 1. These cash-flows represent *real* expected pension payments, to which a cumulative inflation factor should be added so as to obtain the actual liability payment.¹⁴ The duration of the pension fund liability is the maturity of the indexed zero-coupon bond that has the same sensitivity to interest rates as the coupon bond that models the liability. The present value at time 0 of all future payments is given by:

$$\sum_{t_i=1}^{75} n_{t_i} I(0, t_i)$$

The duration τ_0 is defined by:

$$-\frac{1 - e^{-a\tau_0}}{a} \frac{1}{\sum_{t_i=1}^{75} n_{t_i} I(0, t_i)} \sum_{t_i=1}^{75} n_{t_i} I(0, t_i) - \frac{1 - e^{-at_i}}{a} \quad (5.1)$$

¹⁴In practice, inflation indexation is sometimes conditional, with indexation conditions that can be complex and typically depend on the funding ratio of the pension fund and the inflation rate, combined with a minimum and maximum level of indexation. We shall assume away this additional complexity in the empirical exercise that follows, and consider for simplicity a full indexation payment.

Numerically, we get that $\tau_0 = 11.32$ years

5.1 Methodology

We first use as a base case empirical exercise the very model introduced in Section 2 to generate stochastic scenarios for all state variables involved, namely interest, inflation and equity risk factors, and characterize the probability distribution of the terminal funding ratio according to various optimal strategies, and for different choices of the time-horizon and risk-aversion parameters. This allows us to assess the performance of the competing allocation policies, not only in terms of expected utility, but also in terms of terminal funding ratio distribution as well as various heuristic risk and performance indicators used in actual pension fund practice, such as the probability and magnitude of a shortfall. In this subsection, we only need the optimal payoffs of the various strategies, and these payoffs are the same in the general case as in the zero-coupon case. Hence, for realism purposes, we assume that several liability payments take place before the horizon T of the pension fund.

With no loss of generality, we assume that the investment opportunity set includes a single stock index, regarded as an efficient combination of individual stocks, in addition to a zero-coupon bond and an inflation-indexed bond with maturity corresponding to the duration of pension payments. Our base case parameters, summarized in Table 2, are mostly taken from Munk et al. (2004), who also model the nominal interest rate as an Ornstein-Uhlenbeck process and the price index as a Geometric Brownian motion.¹⁵ For each set of parameters values, $N = 5000$ optimal terminal asset values are obtained for each relevant strategy. In all cases, we have assumed that the pension fund was initially fully funded, that is $A_0 = L_0 = \sum_{t_i=1}^{75} n_{t_i} I(0, t_i)$.

5.2 Numerical Results

In Tables 4 and 5, we seek to compare the mean value of the terminal funding ratio when only a lower bound $k = 90\%$ is imposed to the mean value when both a lower and an upper bounds $k = 90\%$ and $k' = 110\%$ are imposed. In Table 3, where no lower bound is imposed, we also compute this conditional mean given that the terminal funding ratio lies between k and k' . The Tables also report the shortfall probability, defined as the probability that pension fund will have a deficit at horizon date, that is $\mathbb{P}(A_T^* < L_T)$ and a relative expected shortfall relative,

¹⁵The main difference between their model and ours is that they also assume an Ornstein-Uhlenbeck process for the expected inflation rate, whereas we take it as a constant, which we assume is given by the long-term mean level used by Munk et al. (2004). Beside, Munk et al. (2004) provide an estimate for the market price of interest rate risk, but neither for the market of inflation risk nor for the market price of equity risk. We set the former at 0, and we set the latter to the value found used in Brennan and Xia (2002), namely 0.343.

defined as the average size of the relative deficit when in deficit: $\mathbb{E} \left[1 - \frac{A_T^*}{L_T} \mid A_T^* < L_T \right]$.

In Table 3 and Figure 1, we provide information regarding the distribution of the final funding ratio when no (implicit or explicit) constraints are introduced. Parameters are fixed at their base case values (see Table 2), except for the risk-aversion parameter γ which takes on the values 2, 5 and 10, and the investment horizon T , which takes on the values 1, 10, 20 and 50. As expected, we find that the dispersion of the final funding ratio increases with T and decreases with γ . Indeed, a lower risk-aversion parameter implies a higher investment in the performance-seeking portfolio, and hence a higher performance potential coupled with a higher funding risk. On the other hand, for a given risk-aversion parameter value, we find that the range of funding ratio values increases with the time-horizon T as more time is allowed for uncertainty to play a role. Even for $\gamma = 10$, we find that the minimum funding ratio obtained is lower than 90% for a one-year horizon, and lower than 60% for a 50 year horizon. This provides justification for the introduction of funding ratio constraints aiming at imposing a left-truncation of the final funding ratio distribution. More importantly perhaps, we find that the average relative deficit can be rather sizable. It is for example equal to 28% when $T = 20$ and $\gamma = 2$. Similarly, we find that maximum funding ratio values can be very high, especially for long horizons and low risk aversion.

In Table 4 and Figure 2, we test for the introduction of explicit constraints, with a minimum kept at 90%. In this case, we find that the minimum is reached, and it is reached with a relatively high probability, confirming the fact that the margin for error is more fully utilized with these strategies compared to the strategies with implicit constraints. In fact, the dispersion of the funding ratio distribution is wider on both sides when the strategy with explicit constraints is implemented, a standard risk-return trade-off. Hence, when $T = 20$ and $\gamma = 5$, the maximum reached is higher than 3 in the explicit constraint case while it reaches a mere 1.3 value in the implicit constraint case. Again, we find that downside protection comes at the cost of a more limited access to the upside potential, as can be seen from the fact that the expected terminal funding ratio conditional upon being larger than the constraint k , $\mathbb{E}[F_T^* \mid F_T^* \geq k]$, is always strictly lower in the constrained case compared to the unconstrained case. For example, when $T = 20$ and $\gamma = 5$, the conditional mean reaches 1.17 in the (explicitly) constrained case, while it is 1.51 in the unconstrained case, and 1.05 in the case with implicit constraints.

Finally, in Table 5 and Figure 3, we introduce an additional upper bound constraint, with a maximum funding ratio value set at $k' = 110\%$. Giving up access to the upside potential above 110% allows one to decrease the cost of downside protection, as can be seen from the fact that the average of terminal funding ratio values conditional upon being in the range between 90% and 110% are higher when the upper bound is introduced compared to when it is not introduced. In fact, focusing again on $T = 20$ and $\gamma = 5$, we have that $\mathbb{E}[F_T^* \mid k \leq F_T^* \leq k'] =$

1.04 when both constraints are imposed, while it merely reached 0.95 when only the lower constraint was imposed. Comparing the solution with both constraints to the unconstrained case from Table 3, we find in the unconstrained case that $\mathbb{E}[F_T^*|k \leq F_T^* \leq k'] = 1.01 < 1.04$. Hence, the addition of the short option position allows for an increase in the mean funding ratio on the range of values between 0.9 and 1.1.

6 Conclusion and Extensions

Defined-benefit pension funds are currently facing a serious challenge and dilemma. On the one hand, the desire to alleviate the burden of contributions leads them to invest significantly in equity markets and other classes poorly correlated with liabilities but offering superior long-term performance potential. On the other hand, stricter regulatory environments and accounting standards give them more incentive to invest a dominant fraction of their portfolios in assets highly correlated to liabilities. The point of our paper is not to question whether minimum funding ratio constraints are desirable or not, but instead to try and provide an analysis of the optimal investment policy given such constraints. When funding ratio constraints are explicitly accounted for, the optimal policies, for which we obtain analytical expressions, are shown to extend standard Option-Based Portfolio Insurance (OBPI) strategies to a relative (with respect to liabilities) risk context. We also show that the introduction of maximum funding ratio targets would allow pension funds to decrease the cost of downside liability risk protection while giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero.

This analysis suffers from a number of caveats. First, it has to be acknowledged that the optimal payoffs presented in this paper are not attainable using standard exchange-traded option contracts. In fact, they rely on complex exchange options involving the liability hedging portfolio as well as (some non trivial function of) the performance-seeking portfolio, which could only be found in the form of over-the-counter products. An interesting question, which we leave for further research, would consist in analyzing whether it would be possible to provide a reasonably good approximation of the optimal payoff with a (dynamic) position in a portfolio of plain vanilla products. Also, we have focused on the pension fund situation, mostly taken in isolation from the sponsor company. In fact, we have adopted the perspective of the pension fund manager, who has been assumed to have preferences over terminal funding ratio of the pension fund. Abstracting away from the sponsor company allows for a dramatic simplification of the problem, and it would be desirable to develop a more integrated approach to asset-liability management, with a focus on optimal allocation and contribution policies from the shareholders' standpoint. This would however require a formal capital structure model, with an analysis of the rational valuation of liability streams as defaultable claims,

as well as an analysis of the impact of asset allocation decisions on the sponsor company credit ratings. It would also require a careful analysis of the agencies issues amongst the various stakeholders, including equity holders and bondholders of the sponsor company, but also workers and pensioners, as well as managers of the pension funds and their trustees. These questions are left for further research.

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A Prices of the Zero-Coupon Bonds

The expression for the nominal zero-coupon bond is standard (see e.g. Vasicek (1977)):

$$B(t, \tau_1) = e^{\alpha(\tau_1-t)r_t + \beta_1(\tau_1-t)}$$

where:

$$\begin{aligned}\alpha(s) &= -\frac{1 - e^{-as}}{a} \\ \beta_1(s) &= -\tilde{b}s + \tilde{b}\frac{1 - e^{-as}}{a} + \frac{\|\sigma_r\|^2}{2a^2} \left[s - 2\frac{1 - e^{-as}}{a} + \frac{1 - e^{-2as}}{2a} \right]\end{aligned}$$

The price of the indexed zero-coupon bond follows from the equalities:

$$\begin{aligned}\Phi_{\tau_2} &= \Phi_t \exp \left[\int_t^{\tau_2} \left(\tilde{\varphi} - \frac{\|\sigma_\Phi\|^2}{2} \right) du + \int_t^{\tau_2} \sigma'_\Phi d\hat{z}_u \right] \\ \int_t^{\tau_2} r_u du &= \tilde{b} \left[\tau_2 - t - \frac{1 - e^{-a(\tau_2-t)}}{a} \right] + \frac{1 - e^{-a(\tau_2-t)}}{a} r_t + \frac{1}{a} \int_t^{\tau_2} (1 - e^{-a(\tau_2-u)}) \sigma'_r d\hat{z}_u\end{aligned}$$

and from standard computations of expectations of log-normal random variables.

Applying Ito's lemma, we get the volatility vectors σ_B and σ_I :

$$\begin{aligned}\sigma_B(t, \tau_1) &= \alpha(\tau_1 - t)\sigma_r \\ \sigma_I(t, \tau_2) &= \alpha(\tau_2 - t)\sigma_r + \sigma_\Phi\end{aligned}$$

B Proof of Proposition 2

The first-order optimality condition reads:

$$\frac{1}{L_T} \left(\frac{A_T^{*k}}{L_T} \right)^{-\gamma} - \nu_1 M_T + \frac{\nu_2}{L_T} = 0$$

where $\nu_2 \geq 0$, $\nu_2 \left(\frac{A_T^{*k}}{L_T} - k \right) = 0$ and $A_T^{*k} \geq kL_T$. This implies that:

$$\frac{A_T^{*k}}{L_T} = kL_T + \left[L_T (\nu_1 M_T L_T)^{-\frac{1}{\gamma}} - kL_T \right]^+$$

The optimal net wealth process is thus given by:

$$A_t^{*k} = kL_t + \frac{1}{M_t} \mathbb{E}_t \left[\left[(\nu_1 M_T L_T)^{1-\frac{1}{\gamma}} - kM_T L_T \right]^+ \right]$$

which can also be written as:

$$A_t^{*k} = kL_t + \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_u du} (\xi A_T^{*u} - kL_T)^+ \right] \quad (\text{B.1})$$

where A_T^{*u} is the optimal final net wealth in the absence of constraints on the funding ratio, ξ is equal to $(\nu_1/\nu)^{-\frac{1}{\gamma}}$ and ν is the Lagrange multiplier associated with the budget constraint in (2.10). The expectation in (B.1) is thus the price at time t of an exchange option of maturity T between a portfolio of terminal value A_T^{*u} and a portfolio of terminal value kL_T .

We can apply Margrabe's formula to find the price of the exchange option:

$$A_t^{*k} = kL_t + \xi A_t^{*u} \mathcal{N}(d_{1,t}) - kL_t \mathcal{N}(d_{2,t})$$

where $d_{1,t}$ and $d_{2,t}$ are given in the statement of the proposition. We then apply Ito's lemma to get:

$$\begin{aligned} dA_t^{*k} = & (\dots) dt + kL_t \boldsymbol{\sigma}_I(t, T_0)' d\mathbf{z}_t + (A_t^{*k} - kL_t) \boldsymbol{\sigma}_I(t, T_0)' d\mathbf{z}_t \\ & + \frac{\xi A_t^{*u} \mathcal{N}(d_{1,t})}{\gamma} [\boldsymbol{\lambda} - \boldsymbol{\sigma}_I(t, T_0)]' d\mathbf{z}_t \end{aligned}$$

Writing $\xi A_t^{*u} \mathcal{N}(d_{1,t})$ as $A_t^{*k} - kL_t \mathcal{N}(-d_{2,t})$ and rearranging terms lead to the optimal portfolio strategy written in the Proposition.

C Proof of Proposition 5

Using the notation of Propositions 2 and 4, we have that:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} (\xi A_T^{*u} - kL_T)^+ \right] \\ = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} (\xi' A_T^{*u} - kL_T)^+ \right] - \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_t dt} (\xi' A_T^{*u} - k' L_T)^+ \right] \end{aligned}$$

Hence we obtain that the price of the option to exchange kL_T for ξA_T^{*u} is strictly lower than the price of the option to exchange kL_T for $\xi' A_T^{*u}$. Given that the price of an option to exchange a payoff for another payoff is an increasing function of the present value of the latter payoff, we finally have:

$$\xi(A_0 - L_0) < \xi'(A_0 - L_0)$$

which implies that $\xi < \xi'$. The proposition follows from this inequality.

D Equivalence of (4.1) and (4.3)

We first compute the Fenchel transform of $\tilde{u}^{k,k'}$, which is defined as:

$$v(y) = \sup_{x \in \mathbb{R}} [\tilde{u}^{k,k'}(x) - xy]$$

Standard univariate maximization leads to:

$$\begin{aligned} v(y) &= \frac{\gamma}{1-\gamma} y^{1-\frac{1}{\gamma}} && \text{if } k'^{-\gamma} \leq y \leq k^{-\gamma} \\ v(y) &= \frac{k^{1-\gamma}}{1-\gamma} - ky && \text{if } y > k^{-\gamma} \\ v(y) &= \frac{k'^{1-\gamma}}{1-\gamma} - k'y && \text{if } 0 < y < k'^{-\gamma} \end{aligned}$$

Hence v is continuously differentiable on the half line $(0, \infty)$, and we have that:

$$v'(y) = -k \mathbb{1}_{\{y^{-\frac{1}{\gamma}} < k\}} - y^{-\frac{1}{\gamma}} \mathbb{1}_{\{k \leq y^{-\frac{1}{\gamma}} \leq k'\}} - k' \mathbb{1}_{\{y^{-\frac{1}{\gamma}} > k'\}}$$

(4.3) is a maximization program involving a non-smooth utility function, which leads us to apply one of the existence results stated in Schachermayer (2001) or in Bouchard et al. (2004). Theorem 3.1 in Bouchard et al. (2004) shows that the solution to (4.7) is given by:¹⁶

$$F_T^* = -v'(\nu M_T)$$

or, equivalently:

$$F_T^* = k + \left[(\nu M_T)^{-\frac{1}{\gamma}} - k \right]^+ - \left[(\nu M_T)^{-\frac{1}{\gamma}} - k' \right]^+$$

This is exactly the same optimal terminal funding ratio as in Proposition 4.

¹⁶Strictly speaking the existence theorem in Schachermayer (2001) or in Bouchard et al. (2004) require a utility function taking values on the real line, which is not the case here. We are not aware of any extension of these results to functions taking infinite values as is the case here.

Table 1: Schedule of annual liability payments expressed in real terms.

Year	Payment	Year	Payment	Year	Payment	Year	Payment
1	6891.04	21	4620.24	41	1114.46	61	52.1
2	7080.01	22	4422.07	42	1008.22	62	40.86
3	7086.14	23	4233.09	43	908.11	63	32.69
4	7034.05	24	4043.1	44	814.14	64	25.54
5	6980.93	25	3822.45	45	727.31	65	19.41
6	6900.23	26	3598.74	46	646.61	66	15.32
7	6767.44	27	3383.21	47	572.04	67	11.24
8	6704.1	28	3173.8	48	503.6	68	8.17
9	6631.58	29	2976.65	49	440.27	69	6.13
10	6542.7	30	2785.63	50	383.06	70	4.09
11	6435.45	31	2597.67	51	330.97	71	3.06
12	6285.29	32	2413.8	52	283.98	72	2.04
13	6113.68	33	2240.15	53	242.1	73	1.02
14	5940.02	34	2074.67	54	205.32	74	1.02
15	5754.11	35	1914.29	55	172.63	75	1.02
16	5575.34	36	1761.06	56	144.03	76	0
17	5393.52	37	1616.01	57	119.52	77	0
18	5195.35	38	1479.13	58	98.06	78	0
19	5024.76	39	1350.42	59	79.68	79	0
20	4830.67	40	1228.86	60	64.35	80	0

This table presents the schedule of annual pension payments expressed in real terms. The data has been provided for by a Dutch pension fund. The duration of this payment stream, as computed in (5.1), is $\tau_0 = 11.36$.

Table 2: Base case parameters.

Parameter	Estimate
Interest rate process	
a	0.0395
b	0.0369
$\bar{\sigma}_r$	0.0195
Price index process	
φ	0.0357
$\bar{\sigma}_\Phi$	0.0081
Stock value process	
$\bar{\sigma}_S$	0.1468
Correlation parameters	
$\rho_{r\Phi}$	-0.0032
ρ_{Sr}	-0.0845
$\rho_{S\Phi}$	-0.0678
Risk premium parameters	
λ_r	-0.2747
λ_Φ	0
λ_S	0.343

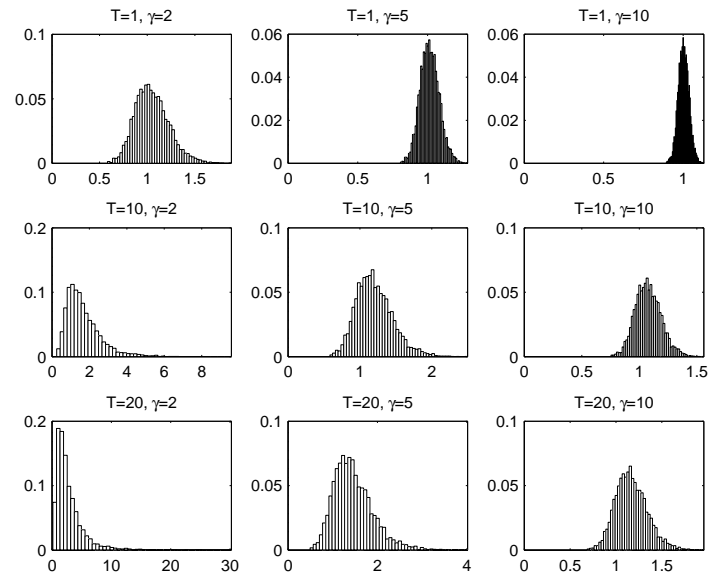
This table contains parameter values for interest rate, price index and stock return processes. These parameter values, as well as the price for interest rate risk, are borrowed from Munk et al. (2004), while the equity risk premium parameter is taken from Brennan and Xia (2002) and the inflation risk premium is set to zero.

Table 3: Distribution of the final funding ratio when utility is from terminal funding ratio and no lower bound is imposed.

$\gamma = 2$			
T	1	10	20
Min	0.58	0.25	0.16
2.5%	0.74	0.52	0.51
25%	0.92	1.03	1.33
50%	1.03	1.46	2.16
75%	1.16	2.1	3.61
97.5%	1.45	4.18	9.37
Max	1.88	9.58	30.22
Mean	1.05	1.68	2.84
St. Dev.	0.18	0.94	2.41
$\mathbb{P}(F_T^{*u} < 1)$	0.42	0.23	0.14
$\mathbb{E}(1 - F_T^{*u} F_T^{*u} < 1)$	0.11	0.24	0.28
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u})$	1.11	1.89	3.12
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.1)$	1	1	1
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.3)$	1.07	1.1	1.1
$\gamma = 5$			
T	1	10	20
Min	0.8	0.58	0.49
2.5%	0.89	0.78	0.79
25%	0.97	1.02	1.15
50%	1.01	1.18	1.4
75%	1.06	1.36	1.72
97.5%	1.16	1.79	2.51
Max	1.29	2.5	4.01
Mean	1.01	1.2	1.46
St. Dev.	0.07	0.25	0.44
$\mathbb{P}(F_T^{*u} < 1)$	0.43	0.22	0.12
$\mathbb{E}(1 - F_T^{*u} F_T^{*u} < 1)$	0.05	0.11	0.13
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u})$	1.02	1.24	1.51
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.1)$	1.01	1.01	1.01
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.3)$	1.02	1.11	1.12
$\gamma = 10$			
T	1	10	20
Min	0.89	0.75	0.69
2.5%	0.94	0.87	0.87
25%	0.98	1	1.05
50%	1	1.07	1.15
75%	1.03	1.15	1.28
97.5%	1.07	1.32	1.55
Max	1.13	1.56	1.96
Mean	1	1.08	1.17
St. Dev.	0.03	0.11	0.17
$\mathbb{P}(F_T^{*u} < 1)$	0.47	0.26	0.16
$\mathbb{E}(1 - F_T^{*u} F_T^{*u} < 1)$	0.03	0.06	0.07
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u})$	1	1.09	1.18
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.1)$	1	1.02	1.02
$\mathbb{E}(F_T^{*u} k \leq F_T^{*u} \leq 1.3)$	1	1.08	1.12

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability, the expected shortfall and the conditional mean of the funding ratio given it lies above $k = 0.9$, or between 0.9 and 1.1, or between 0.9 and 1.3. Parameters are fixed at their base case values (see Table 2). Several liability payments take place and we take $T_0 = 75$ years.

Figure 1: Distribution of the final funding ratio when utility is from terminal funding ratio and no lower bound is imposed.



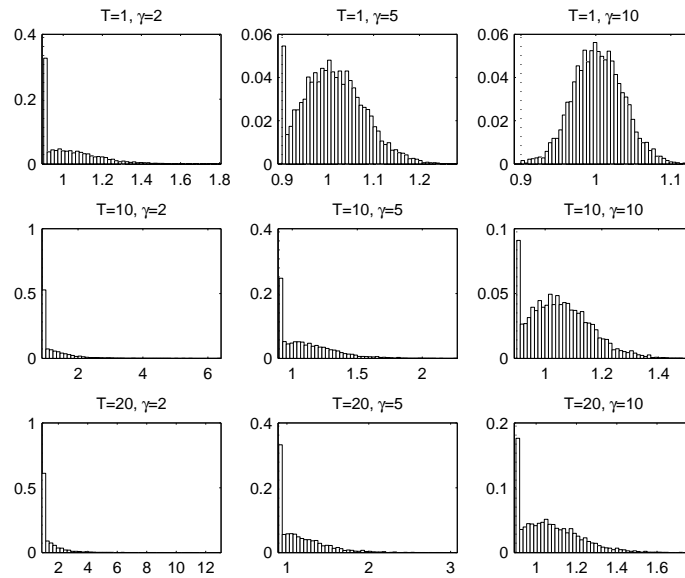
This figure plots the distribution of the optimal terminal funding ratio A_T^*/L_T when utility is from terminal funding ratio and there is no lower bound on this ratio.

Table 4: Distribution of the final funding ratio when utility is from terminal funding ratio and an explicit lower bound is imposed.

$\gamma = 2$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.9	0.9	0.9
50%	0.99	0.98	0.93
75%	1.11	1.4	1.56
97.5%	1.39	2.8	4.04
Max	1.81	6.41	13.04
Mean	1.03	1.24	1.39
St. Dev.	0.14	0.54	0.93
$\mathbb{P}(F_T^{*k} < 1)$	0.52	0.52	0.54
$\mathbb{E}(1 - F_T^{*k} F_T^{*k} < 1)$	0.08	0.09	0.09
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k})$	1.03	1.24	1.39
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.1)$	0.96	0.93	0.92
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.3)$	1.01	0.97	0.95
$\gamma = 5$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.96	0.93	0.9
50%	1.01	1.07	1.07
75%	1.06	1.23	1.32
97.5%	1.16	1.63	1.93
Max	1.28	2.27	3.1
Mean	1.01	1.11	1.16
St. Dev.	0.07	0.21	0.3
$\mathbb{P}(F_T^{*k} < 1)$	0.44	0.37	0.4
$\mathbb{E}(1 - F_T^{*k} F_T^{*k} < 1)$	0.05	0.08	0.08
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k})$	1.01	1.11	1.16
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.1)$	1	0.97	0.95
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.3)$	1.01	1.04	1.02
$\gamma = 10$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.94	0.9	0.9
25%	0.98	0.97	0.95
50%	1	1.05	1.05
75%	1.03	1.12	1.16
97.5%	1.07	1.29	1.41
Max	1.13	1.52	1.78
Mean	1	1.05	1.07
St. Dev.	0.03	0.1	0.14
$\mathbb{P}(F_T^{*k} < 1)$	0.47	0.34	0.37
$\mathbb{E}(1 - F_T^{*k} F_T^{*k} < 1)$	0.03	0.06	0.07
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k})$	1	1.05	1.07
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.1)$	1	1	0.98
$\mathbb{E}(F_T^{*k} k \leq F_T^{*k} \leq 1.3)$	1	1.05	1.05

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability, the expected shortfall and the conditional mean of the funding ratio given that it lies between $k = 0.9$ and 1.1 or 1.3. Parameters are fixed at their base case values (see Table 2). The explicit lower bound k is set to 0.9. Several liability payments take place and we have $T_0 = 75$ years.

Figure 2: Distribution of the final funding ratio when utility is from terminal funding ratio and an explicit lower bound is imposed.



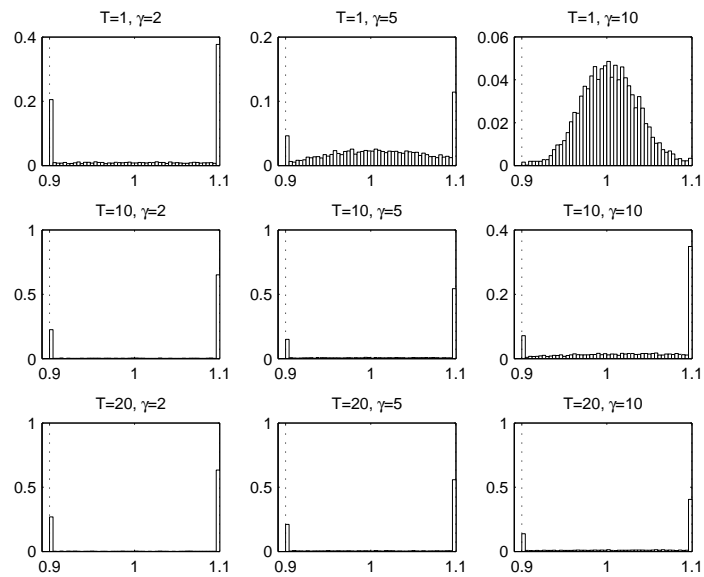
This figure plots the distribution of the optimal terminal funding ratio A_T^*/L_T when utility is from terminal funding ratio and there is an explicit lower bound on this ratio.

Table 5: Distribution of the final funding ratio when utility is from terminal funding ratio and explicit lower and upper bounds are imposed ($k' = 1.1$).

$\gamma = 2$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.93	0.95	0.9
50%	1.04	1.1	1.1
75%	1.1	1.1	1.1
97.5%	1.1	1.1	1.1
Max	1.1	1.1	1.1
Mean	1.02	1.04	1.04
St. Dev.	0.08	0.09	0.09
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.41	0.28	0.31
$\mathbb{E}(1 - F_T^{*k,k'} F_T^{*k,k'} < 1)$	0.07	0.09	0.09
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'})$	1.02	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.1)$	1.02	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.3)$	1.02	1.04	1.04
$\gamma = 5$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.97	0.97	0.95
50%	1.01	1.1	1.1
75%	1.06	1.1	1.1
97.5%	1.1	1.1	1.1
Max	1.1	1.1	1.1
Mean	1.01	1.04	1.04
St. Dev.	0.06	0.08	0.08
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.43	0.29	0.32
$\mathbb{E}(1 - F_T^{*k,k'} F_T^{*k,k'} < 1)$	0.05	0.07	0.08
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'})$	1.01	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.1)$	1.01	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.3)$	1.01	1.04	1.04
$\gamma = 10$			
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.94	0.9	0.9
25%	0.98	0.98	0.96
50%	1	1.05	1.06
75%	1.03	1.1	1.1
97.5%	1.07	1.1	1.1
Max	1.1	1.1	1.1
Mean	1	1.03	1.03
St. Dev.	0.03	0.07	0.08
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.47	0.32	0.34
$\mathbb{E}(1 - F_T^{*k,k'} F_T^{*k,k'} < 1)$	0.03	0.05	0.07
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'})$	1	1.03	1.03
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.1)$	1	1.03	1.03
$\mathbb{E}(F_T^{*k,k'} k \leq F_T^{*k,k'} \leq 1.3)$	1	1.03	1.03

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability and the expected shortfall and the conditional mean of the funding ratio given that it lies between $k = 0.9$ and 1.1 or 1.3 . Parameters are fixed at their base case values (see Table 2). Several liability payments take place and we have $T_0 = 75$ years. The explicit lower bound k is set to 0.9 and the explicit upper bound k' at 1.1 .

Figure 3: Distribution of the final funding ratio when utility is from terminal funding ratio and explicit lower and upper bounds are imposed.



This figure plots the distribution of the optimal terminal funding ratio A_T^*/L_T when utility is from terminal funding ratio and there are explicit lower and upper bounds on this ratio.